

From Hotelling to Nakamoto: The Economic Meaning of Bitcoin Mining

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Abstract

We develop a continuous-time dynamic model for Bitcoin mining from the miners' perspective by borrowing idea of the classic Hotelling model for exhaustible resources. The model is rich enough to incorporate declining Bitcoin rewards and random transaction fees as well as inventory and demand levels, and is able to calibrate to empirical data. We find that high jump risk and transaction fees are major forces driving miners to reduce their inventory even when Bitcoin prices are quite low or very volatile. Our model can also explain why the average transaction fee rate stays flat from 2014 to 2016 and increases dramatically in 2017.

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In the classic exhaustible resource model pioneered by Hotelling (1931), miners maximize their revenue by controlling their production rate. In contrast, the production rate of Bitcoin is predetermined by the Bitcoin system in the form of Bitcoin rewards to miners. On the other hand, Bitcoin miners can control the supply of Bitcoin to users via an inventory management, i.e., a selling strategy. In addition to Bitcoin awards, Bitcoin miners receive transaction fees attached by users. A natural question is to how Bitcoin miners manage their inventory so as to maximize their revenue from Bitcoin awards and transaction fees.

The question is closely related to a stylized fact as shown in Fig. 1: the ratio of the miners' inventory to the total number of mined bitcoins declines steadily, even when Bitcoin prices are low and extremely volatile (see Athey et al., 2016). Given the fact that Bitcoin rewarding is a unique way of Bitcoin supply, Fig. 1 has two implications: (1) miners keep selling their Bitcoins; and more importantly, (2) miners sell more Bitcoins than what they mine during the period of 2013-2015, regardless the drastic fluctuation of Bitcoin prices and the scarcity of Bitcoins.

This paper also attempts to address the stylized fact revealed by Fig. 2: the average transaction fee rate, defined as the average transaction fees paid by users for a unit Bitcoin, seems to be flat from 2014 to 2016 and increases dramatically in 2017. It is also reported in Easley, O'Hara, and Basu (2019) that many users may get their orders processed without paying any transaction costs.

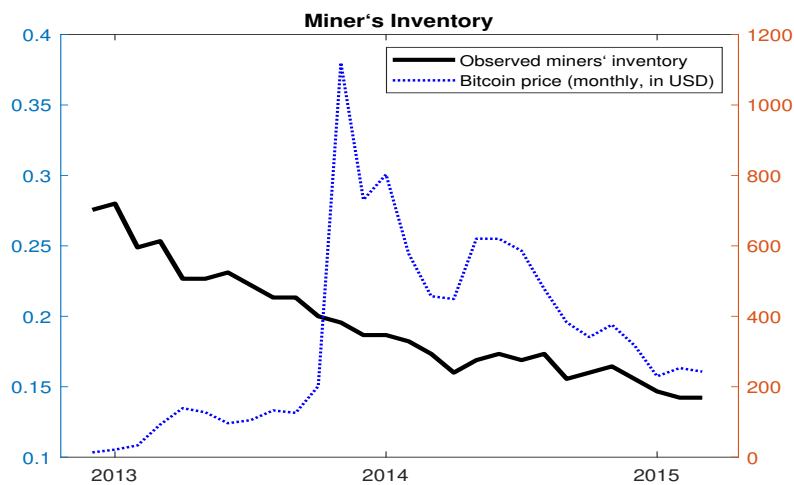


Figure 1: **Miner's inventory proportional to the supply from 2013 to 2015** (Athey et al., 2016).

$$\text{Proportional inventory} = \frac{\text{Miners' aggregate inventory at time } t}{\text{Cumulative Bitcoin supply at time } t}$$

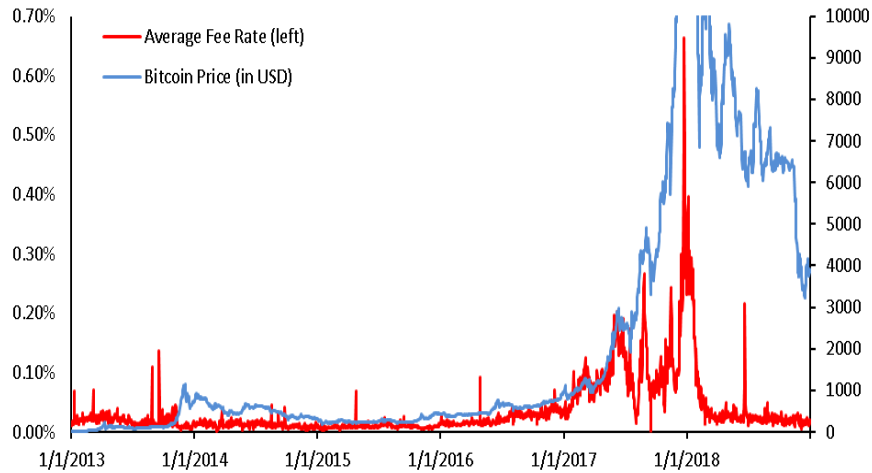


Figure 2: **The dynamics of average transaction fee rate and Bitcoin price from 2013 to 2018.** The Bitcoin price only shows the part below 10000 USD. Average fee rate = $\frac{\text{Miners' transaction fees at time } t}{\text{Validated transaction volume at time } t}$

In this paper, we develop a continuous-time dynamic model for Bitcoin mining by borrowing idea of the classic Hotelling model for exhaustible resources. Our major contributions can be summarized as follows. First, our model is novel. The classic Hotelling model and its extensions are built on a production economy, in which miners control production rate to affect circulation. In contrast, our Bitcoin model introduces miners' inventory as a new state variable, where miners affect Bitcoin circulation via an *inventory* management. Moreover, our model incorporates the partially closed-loop of Bitcoin supply that is missing in traditional resource models: miners, whom the system first distributes Bitcoins as rewards to, sell their holdings to users, while in turn users provide miners with a certain amount of bitcoins as transaction fees; for details, see Section 2. In addition, it should be pointed out that the magnitude of transaction fees is endogenously determined by the market, and we need to first model transaction fees; see Section 2.4. Last but not least, our model can be extended to study the miner's optimal exit choice in Bitcoin mining business; see Appendix B.

Second, our model can calibrate to empirical data and explain the aforementioned two stylized facts about the declining miners' inventory and the average transaction fee rate. Indeed, our model-implied miners' inventory perfectly matches the real data (see Fig. 8), and our model-implied

average fee rate and aggregate transaction fees capture pretty well the dynamics of the observed ones (see Figures 9), which justifies our model.

Third, our model has the following interesting economic implications.

- (i) Miners reduce their inventory to make profits and sustain their mining service. We find that high jump risk is one of major forces driving miners to sell their Bitcoin holdings at an early stage even when Bitcoin prices are quite low or very volatile (see Fig. 11).
- (ii) We find that transaction fees are another driving force for miners to reduce their inventory. The intuition is quite simple. The outstanding volume of Bitcoins significantly affects the magnitude of transaction fees: the larger the volume (or equivalently, the less the miners' inventory), the higher the transaction fees. Moreover, transaction fees attached by users provide a unique source of income to miners after termination of Bitcoin rewards. Therefore, miners have a strong incentive to reduce their inventory for the future consideration of transaction fees; see Fig. 17.
- (iii) The miners' optimal selling strategy characterized by our model indicates that for any fixed inventory level, there exists a threshold demand level, called (optimal) selling barrier, above which one never sells Bitcoins (see Fig. 10). The selling barrier turns out to be increasing with the inventory level, as miners tend to sell Bitcoin earlier for a higher inventory level in order to receive more transaction fees in the future.
- (iv) Our model suggests that a high (low) Bitcoin demand leads to a high (low) transaction fee rate. In particular, we find that when Bitcoin demand is too small to make miners' capacity full, miners may take all orders submitted by users even without any transaction fees, and the average fee rate tends to be invariant as a result; when Bitcoin demand is intensive such that miners' capacity is full, miners will select those orders with larger transaction fees to maximize their profit, which leads the average fee rate to increase drastically. This finding is consistent with market observations (see the periods of 2014-2016 and 2017, respectively, in Figures 2 and 9, and comparative statics in Fig. 16).

(v) Our model examines the importance of block rewards and transaction fees in Bitcoin mining quantifying the miner’s value from time to time. In short run, the miner’s value is dominated by block rewards and decreases dramatically in the design of declining block rewards, while in long run, the miner’s value is dominated by transaction fees and exhibits an increasing form (see Fig. 18).

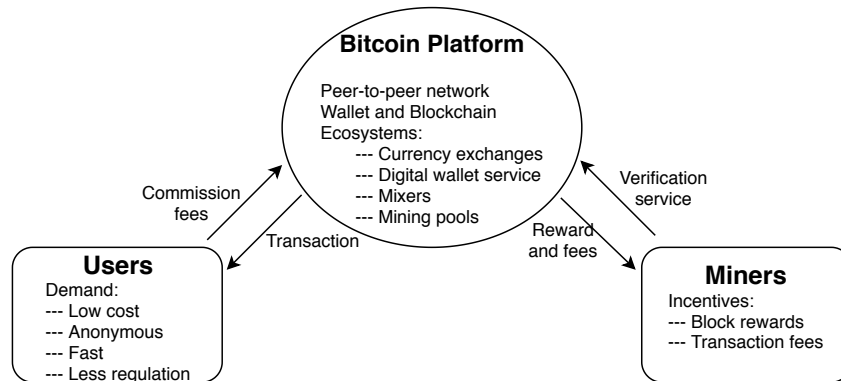


Figure 3: The Bitcoin system

1 Background and Literature Review

1.1 Background of Bitcoin

Bitcoin, developed by an anonymous group (Nakamoto, 2009) in 2009, is a decentralized online payment system with a specially designed crypto-currency. After 10-year’s development, the Bitcoin system has been growing into a huge market with enormous users. The Bitcoin market capital grows to be nearly 100 billion in USD, with the price peak of USD19,000 reached in 2017.

As shown in Fig. 3, there are two types of participants in the Bitcoin system: users and miners. Compared to other traditional payment systems (e.g., visa), the Bitcoin system has several benefits

to users, such as anonymous transaction records, fast processing, low cost of payment, and less regulation. On the other hand, miners play a dispensable role in the system. They are responsible for maintenance operations of the system, including validating transactions and updating the distributed ledger (i.e. the blockchain).

The Bitcoin system has a special incentive setting to compensate the activities of miners. For each transaction order, users place certain transaction fees before the order is picked by some miner into her candidate block based on the attached fees. Once the miner’s candidate block is successfully attached to the blockchain, the miner will get two sources of income: (1) block rewards predetermined by the system; (2) the transaction fees attached to the order. The two sources of

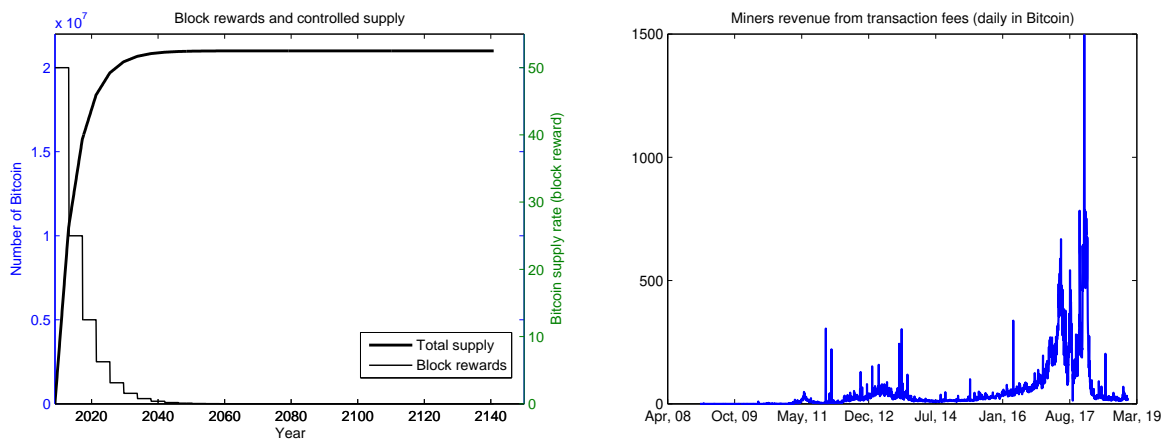


Figure 4: **Miner’s revenue from Bitcoin rewards and transaction fees.** Left figure shows the block rewards for each block, which halves every four years, and cumulative supply of Bitcoin. Right figure shows the daily transaction fees in Bitcoin from 2009 to 2018. Total number of Bitcoins is 21 million. The supply of Bitcoin will end on May 7th, 2140.

income have distinct nature. In fact, block rewards are *deterministic*, *exogenous*, and *scarce* in the sense that the amount of block rewards is halved after every four years and will vanish after 2140; see the left panel of Fig. 4. Hence, the total supply of Bitcoin is fixed and finite, and *scarcity* is one of the most distinct features of Bitcoin, which may explain why Bitcoin is often treated by investors as gold and where the name of miner comes from. In a sharp contrast, transaction fees are *stochastic*, *endogenous*, and *permanent* in the sense that they are determined endogenously by the market and would not die out as long as there are users; see the right panel of Fig. 4.

The unique pattern of block rewards indicates that the lifetime of the Bitcoin system can be divided into two stages: the short-run period (i.e. before the termination of Bitcoin supply), and the long-run period (i.e. after the termination of Bitcoin supply). In long-run, miners' activities are only compensated by attached fees. Therefore, transaction fees will become more and more important to miners.

1.2 Literature review

There are three strands of literature on Bitcoin relevant to our model. The first strand of literature is about resource models, dated back to Hotelling (1931) and later extended along several directions (see, e.g., Stewart (1980), Levhari and Pindyck (1981), Malueg and Solow (1990)). Our work complements this literature by studying Bitcoin as a durable and exhaustible resource and treating miners' mining activity analogous to the extraction activity in natural resource production. However, our Bitcoin model differs from classic resource models primarily in three ways: (1) Inventory is included in our model, while traditional resource models does not involve inventory due to the presence of storage costs. Inventory in Bitcoin is important simply because the storage cost of Bitcoin is zero and the supply of Bitcoin to users is highly related to miners' inventory.¹ (2) We model miners' activities in accumulating transaction fees by taking into account the partially closed-loop of Bitcoin supply. (3) We model Bitcoin demand as a hybrid of "S"-shaped diffusion form, jump risk, and regime switching due to Bitcoin being a novel, non-government backed medium of exchange, while most resource models only assume growing demand.

The second strand of literature focuses on transaction fees in Bitcoin payment. Huberman et al. (2017) consider a simplified model to examine how the system raises revenue to pay for its infrastructure. They show that transaction fees and infrastructure level are determined in an equilibrium of a congestion queuing game. Easley, O'Hara, and Basu (2019) develop a Nash equilibrium model to investigate the role of transaction fees and explain strategic behaviors of

¹The Bitcoin system firstly supplies Bitcoin to miners via distributing block rewards and then miners supply Bitcoin to users by reducing their inventory and charging transaction fees.

miners and users.

Table 1: Literature comparison

Huberman et al. (2017)	One period	Fee paying strategy from users' side
Easley, O'Hara, and Basu (2019)	One period	Nash equilibrium of users' fee paying strategy
Our model	Continuous time dynamic model	Transaction fees from miner's perspective incorporating declined block rewards and miners' inventory

Our paper complements Huberman et al. (2017) and Easley, O'Hara, and Basu (2019) by studying transaction fees with a dynamic stochastic control model from the miners' perspective both before and after the termination of Bitcoin supply. We model transaction fees paid by users in an aggregate way, i.e., a distribution function is used to model the dispersion of transaction fees in submitted orders. On the other hand, the limit of capacity in processing transactions induces miners to pick those orders with higher transaction fees. Such setting enables us to understand the dynamics of average transaction fee rate in Fig. 2 from the miners' perspective. Both Huberman et al. (2017) and Easley, O'Hara, and Basu (2019) study transaction fees with a static model from the users' perspective, either ignoring block rewards or assuming constant block rewards. By contrast, our model is dynamic, incorporate the decline form of block rewards, and can explain how the coexistence of the declined block rewards and the increasing transaction fee income affects miner's value and mining strategies.

The third strand of literature considers Bitcoin as currency. Ron and Shamir (2013) and Athey et al. (2016) analyze the usage of Bitcoin and its value as a currency. Yermack (2013) reviews the history of Bitcoin and the statistical properties of historical Bitcoin prices and argues that Bitcoin does not behave much like a currency according to the criteria widely used by economists, and instead, Bitcoin resembles a speculative investment similar to the Internet stocks of the late 1990s. Bolt and Oordt (2016) treat Bitcoin as a medium of exchange and use the equation of quantity to analyze the exchange rate of virtual currency. Recently, Schilling and Uhlig (2019) provide a model of an endowment economy for Bitcoin and find that its fundamental price forms a martingale. Schilling and Uhlig (2019) extend this analysis to study currency substitution under

transaction costs. Benigno, Schilling, and Uhlig (2019) study competition between cryptocurrency and national currencies with a model of a two-country economy. Other related literature includes Saito (2015), Gandal and Halaburda (2014), Halaburda and Sarvary (2016), Gans and Halaburda (2015), Catalini and Gans (2016), etc.

As with this strand of literature, our model treats Bitcoin as a medium of exchange and uses equation of quantity to price Bitcoin. Unlike existing literature, our model parsimoniously assumes that the quantity of goods and services people are willing to purchase with Bitcoin is related to demand shock of Bitcoin. For simplicity, we assume constant form of joint effect of velocity and quantity of Bitcoin, and the application of equation of quantity implies a linear relationship between Bitcoin price and its demand.

Other literature related to our work covers a wide range of topics. For literature about the diffusion of Bitcoin economy, see Bohme, Christin, Edelman, and Moore (2015); Athey et al. (2016); Dixon (1980); Bass (1969); Mahajan, Muller, and Bass (1990); Bass (2004), etc. For Bitcoin mining in mining pools, see Cong, He, and Li (2018). For learning by investors and predictability of returns in Bitcoin, see Detzel et al. (2018). For adoption and valuation of cryptocurrencies, see Con, Li, and Wang (2019). There is extensive literature about the engineering of Bitcoin. The white paper Nakamoto (2009) introduces Bitcoin and describes the Bitcoin system. Eyal et al. (2016) and Sapirshtein et al (2016) analyze the equilibrium among miners and show that the proper design of the blockchain protocol produces a reliable system in equilibrium if all miners are sufficiently small. Babaioff et al. (2012) analyze the incentives to propagate information in the Bitcoin system. Narayanan et al. (2016) offer an elaborate description and analysis of the system. Croman et al. (2016) provide cost estimates of the Bitcoin system and analyze the potential for transaction throughput. Eyal et al. (2016) suggest an alternative design that aims to construct a system with a higher transaction throughput. Carlsten et al. (2016) analyze how miners' incentive changes when miners are rewarded with transaction fees instead of newly created coins. Chiu and Koepl (2017) evaluate the welfare implications of printing new coins.

2 Model Setup

In this section, we develop a continuous-time model to study the economic meaning of bitcoin mining, where the idea of the classic exhaustible resource model proposed by Hotelling (1931) is borrowed. We attempt to formulate the problem as a control problem: A miner chooses a dynamic selling rate of Bitcoin $\{Q_t\}_{t \geq 0}$ to maximize the expectation of her discounted accumulative profit

$$\int_0^{\infty} e^{-\beta t} (P_t Q_t - C(Q_t, H_t)) dt \quad (1)$$

subject to certain dynamics of Bitcoin price P_t (i.e., exchange rate) and the miner's inventory level H_t to be specified later. Here $\beta > 0$ is a discount factor, $P_t Q_t$ represents the revenue flow, and $C(Q_t, H_t)$ is the cost flow depending on the selling rate Q_t as well as the inventory level H_t . A specific form of function $C(\cdot, \cdot)$ will be given later.

As a comparison, the classic Hotelling model uses the production rate as control variable, and involves the reserve level of an exhaustible resource other than the inventory level. Another key difference of our model from Hotelling's is that the miner's Bitcoin inventory can be replenished through transaction fees attached by users even after the end of block rewards, whereas the reserve level of an exhaustible resource, which may increase due to exploration, will eventually decrease to zero due to resource scarcity.

For a complete model, the dynamics of two state variables P and H must be specified. In a general equilibrium model, both of the state variables have to be determined *endogenously*, which, however, is intractable and thus is beyond the scope of this paper. In this paper, we consider a partial equilibrium, in which the Bitcoin price dynamics is *exogenously* given, but, the dynamics of the miner's inventory is affected by the miner's selling rate and is thus determined *endogenously*. In what follows, we will give these dynamics in detail.

2.1 Price and demand

The determination of Bitcoin price is challenging. Let us only focus on the role of Bitcoin as a medium of exchange. Hence, we assume that Bitcoin price is determined by the following quantity equation of medium of exchange (Bolt and Oordt, 2016; Fisher (1911); Friedman (1973)):

$$P_t = \theta_p X_t \quad \text{for } t \geq 0, \quad (2)$$

where θ_p is a positive constant decided by Bitcoin supply and velocity, and X_t is a demand factor to be specified later. Another possible interpretation of our assumption comes from the basic economic principle that price is determined by *both* demand *and* supply and the fact that supply is fixed a priori in Bitcoin system.

By the demand factor X_t , we mean the quantity of goods and services that people are willing to purchase with Bitcoin. We assume that it follows a stochastic process

$$dX_t = \mu(X_t, i_t)dt + \sigma(X_t, i_t)dW_t - X_{t-}d\left(\sum_{i=1}^{N_t}(1 - Z_i)\right) \quad (3)$$

where W_t is a standard Brownian motion, N_t is a Poisson process with intensity λ_J , $\mu(\cdot, \cdot)$ and $\sigma(\cdot, \cdot) \geq 0$ stand for the adoption term and volatility term, respectively, $1 - Z \in (0, 1)$ refers to the proportional downward-jump size,² and $i_t \in \{\mathbb{H}, \mathbb{L}\}$ represent two transaction states that correspond to High-active and Low-active markets, respectively, with transition intensities $\zeta = (\zeta_{\mathbb{H}}, \zeta_{\mathbb{L}})$. Borrowing the idea from marketing science, the adoption of Bitcoin can be characterized by the classic diffusion model (e.g. Dixon, 1980; Bass, 1969; Mahajan, Muller, and Bass, 1990; Bass, 2004), which is the so-called ‘‘S’’ shaped demand curve. Particularly, we use Gompertz model with stochasticity in our work (Dixon, 1980; Bass, 2004). Therefore, for a given transaction state $i \in \{\mathbb{H}, \mathbb{L}\}$, the functions $\mu(\cdot, i)$ and $\sigma(\cdot, i)$ take the following forms, respectively:

$$\mu(x, i) = \kappa_i(\nu_i - \ln x)x, \quad \sigma(x, i) = \sigma_i x, \quad (4)$$

²In general, $Z \in (0, 1)$ could be a random variable. Here we assume constant jump size for simplicity.

where $\nu_i > 0$ is called log carrying capacity, standing for the long-run mean of logarithm of demand factor, κ_i called adoption speed, measuring the speed of mean-reverting, and $\sigma_i > 0$ is the volatility. For simplicity, we assume that κ_i and ν_i are independent of market regime, i.e. $\kappa_i \equiv \kappa, \nu_i \equiv \nu$ for $i \in \{\mathbb{H}, \mathbb{L}\}$.

2.2 Block reward and miner's inventory

We consider an economy with identical miners that are endowed with the same computation power ω and the same mining cost. Let ω TH/s be the computing power of a representative miner in terms of hash rate, and denote by π the probability that the miner can successfully mine a block and win rewards/fees in 10 minutes. The winning probability π is approximate to, according to Hayes (2017),

$$\pi = \frac{\omega}{D \times 2^{23}/600}, \quad (5)$$

for the difficulty level D , where $D \times 2^{32}/600$ is called the network hash rate of Bitcoin system .

The dynamics of the miner's Bitcoin inventory is

$$dH_t = (b_t + I_t)\pi dt - Q_t dt. \quad (6)$$

where b_t represents the miner's block reward in a unit of period from successful block mining, I_t represents total transaction fees in the miner's candidate blocks for the same unit of period, and Q_t is the selling rate of the miner at time t . At any time period $[t, t + dt]$, the miner will receive rewards with amount of $(b_t + I_t)\pi dt$ at a probability π . Meanwhile, the miner has the right to sell Bitcoins at a controllable rate $Q_t \geq 0$ from her holdings.

Since Bitcoin supply path is fixed and predetermined, the supply rate b_t is a deterministic and decreasing function of time t , i.e., $b_t = b(t)$. Moreover, $b(t) \equiv 0$ for all $t \geq T$, where T is some predetermined time. So, the cumulative supply at time t is given by

$$S(t) = \int_0^t b(u) du, \quad (7)$$

and the total available supply of Bitcoin is $\int_0^T b(u)du = \bar{S} < \infty$. Moreover, the miner's holding cannot exceed the cumulative supply of Bitcoin, i.e. $0 \leq H_t \leq S(t)$, and short selling is not permitted.

Since transaction fees I_t are voluntarily attached by users, their modeling is not straightforward and is introduced in section 2.4.

2.3 Miner's cost

During the mining and selling process, miner's costs could be incurred from two possible sources: (a) mining cost; and (b) liquidation cost. The mining cost comes from competing and validating blocks. Owing to running mining machines, the primary cost is from the expense of electricity, which is assumed to be a constant C_m , consistent with Easley, O'Hara, and Basu (2019) and Cong, He, and Li (2018). The liquidation cost includes losses due to price drop and losses of marginal utility upon sale. Parsimoniously, as in the investment literature for adjustment cost or execution cost (see, e.g. Hayashi, 1982; Graewe and Horst, 2017), the liquidation cost is assumed to be convex in selling, and is a decreasing function of miner's holding. For analytical tractability, we adopt the following form:

$$C(Q_t, H_t) = \psi P_t Q_t^2 / H_t, \quad (8)$$

where ψ is a parameter.³ Note that the liquidation cost is relate to the inventory level because Bitcoin can be viewed as a special art product with certain scarcity: the lower the inventory, the higher the liquidation cost could be and the less likely the miner is willing to sell her holding. Particularly, when a miner's inventory is zero, i.e. $H = 0$, the marginal dis-utility is infinity. It is also worthwhile pointing out that the cost function is homogeneous of degree one in Q and H , i.e., $C(\delta Q, \delta H) = \delta C(Q, H)$ for $\delta > 0$, which plays a critical role in simplifying calibration.

³The total cost function could be $C_m + C(Q_t, H_t) = C_m + \psi P_t Q_t^2 / H_t$. Since we do not impose budget constraint for simplicity, the constant mining cost C_m does not affect the miner's decision and constraint. Without loss of generality, we can assume $C_m = 0$ and then focus on the liquidation cost $C(\cdot, \cdot)$ only.

2.4 Modelling Transaction Fees

We shall clarify the mechanism of transaction fees, i.e. I_t in (6), in this subsection. Transaction fees are an important source of compensation to miners for their mining activities in the Bitcoin system. In this section, we model transaction fees in the Bitcoin system from the miners' perspective. More specifically, we shall model transaction fees as a result of an optimization problem faced by miners: due to capacity constraint, miners prefer those submitted transaction orders with higher attached fees to maximize their compensation income.⁴

There are three assumptions for transaction fees in our model. First, the Bitcoin system stipulates that a miner's processing of transaction orders confronts a capacity limitation. That is, only a fixed number of orders could be processed by a miner in a unit period of time.⁵ For simplicity, we denote the miner's capacity for a unit period to be G . Second, a miner faces an effective volume of submitted orders, denoted as L . In fact, L is associated with the orders submitted for processing which are stored in the miner's node. As pointed by Biegel (2019), due to the different storage limit of each node, L could be quite different for different miners, which we will elaborate later. Third, we further assume that for all submitted orders, there exists a stable distribution of attached fee rate ϕ , with the density function $f(\phi)$ over the interval $[0, \bar{\phi}]$, where $\bar{\phi}$ is the upper bound of the fee rate decided exogenously by the competition of other payment methods.

Hence, during a unit period, the miner's transaction fees I can be formulated as the following optimization problem:

$$I = \max_{\Phi \in [0, \bar{\phi}]} K(\Phi)L \quad \text{subject to} \quad k(\Phi)L \leq G, \quad (9)$$

⁴At the beginning of each round of mining, a miner will construct a candidate block by selecting orders from the submitted orders stored in her node. Different node could have different storage of the unconfirmed transaction orders (see, e.g. Biegel, 2019). Usually, miners sort the orders with respect to the magnitude of attached fees in mempools and collect the orders into a candidate block as much as they can until the capacity of the candidate block is reached.

⁵In the Bitcoin system, each block has a limitation of capacity (i.e. 1MB), which is about 4000 transaction orders. Since every ten minutes only one block can be successfully validated and attached to the blockchain, for a unit period (e.g. one day), there is a fixed number of orders that can be processed.

where for one unit volume of submitted orders,

$$k(\phi) = \int_{\phi}^{\bar{\phi}} f(s)ds \quad \text{and} \quad K(\phi) = \int_{\phi}^{\bar{\phi}} f(s)sds, \quad (10)$$

and $k(\phi), K(\phi)$ represent the proportion of and transaction fees in selected orders with fee rate no less than ϕ , respectively.

It is easy to see that the optimal choice of threshold fee rate, as a function of L , is given by

$$\Phi^*(L) = \begin{cases} F^{-1}\left(1 - \frac{G}{L}\right), & \text{if } L > G, \\ 0 & \text{if } L \leq G, \end{cases} \quad (11)$$

where $F(\cdot)$ is the cumulative distribution function of $f(\cdot)$, $F^{-1}(\cdot)$ stands for the inverse function of $F(\cdot)$, and G is a universal constant of miner's capacity for a unit of period; see Fig. 5 for illustration.

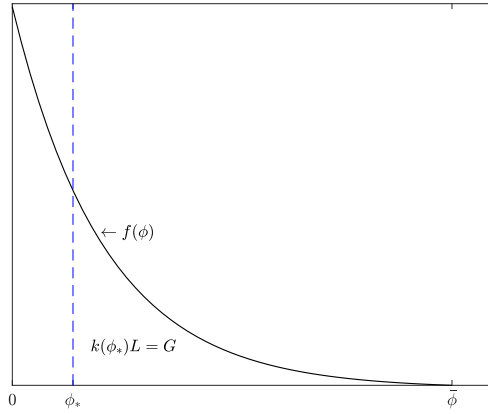


Figure 5: **Distribution of Bitcoin transaction fees and the threshold fee rate .**

For analytical tractability, we need to specify the distribution of attached fee and the volume of submitted orders. Regarding the fee distribution, we choose Beta distribution, i.e.

$$f(y) = \frac{\bar{\phi}^{1-a-b} y^{a-1} (\bar{\phi} - y)^{b-1}}{B(a, b)},$$

where $B(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du$ is the Beta function, and a and b are constants to be calibrated

with market data. It remains to specify a particular function form for the effective volume of submitted orders L . In practice, L is affected by many factors. In this paper, we assume that L depends on the state i and is an increasing function of $(S - H)$, Bitcoin holding by users, and demand level X . In particular, we choose the following special form for L :

$$L(H, X, i; S) = \theta_i(S - H) \log(1 + X), \quad i \in \{\mathbb{H}, \mathbb{L}\} \quad (12)$$

where θ_i is a positive constant.

2.5 Miner's optimization problem

A strategy $\{Q_t\}_{t \geq 0}$ is admissible if (i) for any $t > 0$, $Q_t \geq 0$ is adapted to information filtration up to time t ; and (ii) the resulting inventory level must be non-negative, i.e. $0 \leq H_t \leq S(t)$ for all $t > 0$. Denote by \mathcal{A}_0 the set of all admissible strategies. The mining problem is to choose an admissible strategy $\{Q_t\}_{t \geq 0} \in \mathcal{A}_0$ so as to maximize the expected present profit (1) subject to (2)–(8). This is a standard stochastic control problem.

3 Theoretical Analysis

In this section, we conduct theoretical analysis.

3.1 HJB equation and optimal strategy

To study the miner's optimization problem, we introduce the associated value function as follows:

$$V_i(t, x, h) = \sup_{\{Q_u\}_{u \geq t} \in \mathcal{A}_t} \mathbb{E}_t \left[\int_t^\infty e^{-\beta(u-t)} (P_u Q_u - C(Q_u, H_u)) du \right], \quad (13)$$

where \mathbb{E}_t denotes the expectation conditional on the information up to time t , and \mathcal{A}_t represents the set of admissible strategies starting from $(t, i_t, X_t, H_t) = (t, i, x, h) \in (0, \infty) \times \{\mathbb{H}, \mathbb{L}\} \times (0, \infty) \times [0, S(t)]$.

Characterization by HJB equation

By the dynamic programming principle, the value function $V_i(t, x, h)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation ⁶,

$$\frac{\partial V_i}{\partial t} + \mathcal{L}V_i + \max_{q \geq 0} \left\{ \left[\left(b(t) + K(\Phi^*(L))L \right) \pi - q \right] \frac{\partial V_i}{\partial h} + Pq - C(q, h) \right\} + \mathcal{J}V_i = \beta V_i \quad (14)$$

for $(x, h) \in (0, \infty) \times [0, S(t)]$ and $i \in \{\mathbb{H}, \mathbb{L}\}$, where $L = L(X, H, i; S(t))$, $P = \theta_p x$, $K(\cdot)$ and $\Phi(\cdot)$ are as given in (10) and (11), respectively, and differential operators \mathcal{L} and \mathcal{J} are defined respectively by

$$\mathcal{L}V_i = \frac{1}{2} \sigma^2(x, i) \frac{\partial^2 V_i}{\partial x^2} + \mu(x, i) \frac{\partial V_i}{\partial x}, \quad (15)$$

$$\mathcal{J}V_i = \lambda_J \left[V_i(t, Zx, h) - V_i(t, x, h) \right] + \zeta_i \left[V_i(t, x, h) - V_i(t, x, h) \right] \quad (16)$$

with $\iota \in \{\mathbb{H}, \mathbb{L}\}$ and $\iota \neq i$.

Next we discuss the case after the termination of block rewards. Note that when the system supply is over, i.e., $t \geq T$, $b(t) \equiv 0$ and $S(t) \equiv \bar{S}$, the value function defined in (13) is independent of time t , i.e., $V_i(t, x, h) = V_i(T, x, h) =: V_i^T(x, h)$ for $t \geq T$ and the HJB equation (14) becomes

$$\mathcal{L}V_i^T + \max_{q \geq 0} \left\{ \left(K(\Phi^*(L))L \pi - q \right) \frac{\partial V_i^T}{\partial h} + Pq - C(q, h) \right\} + \mathcal{J}V_i^T = \beta V_i^T \quad (17)$$

for $(x, h) \in (0, \infty) \times [0, \bar{S}]$ and $i \in \{\mathbb{H}, \mathbb{L}\}$, where $L = L(X, H, i; \bar{S})$.

Proposition 1 indicates that in the absence of transaction fees, the mining problem is homogeneous regarding computing power and inventory.

Proposition 1. *Let $V_i(t, x, h; \pi)$ be the value function as defined in (13) with winning probability π . Assume no transaction fees. Then $V_i(t, x, h; \pi)$ is homogeneous of degree one in h and π , i.e., $V_i(t, x, \delta h; \delta \pi) = \delta V_i(t, x, h; \pi)$ for $\delta > 0$.*

⁶A verification theorem that solution of HJB equation (14) equals the value function (13) can be rigorously proved following standard procedures in stochastic control literature, i.e. Cuoco and Liu (2000).

Proof. See proof in Appendix A.1. □

Since transaction fees have been relatively small compared with to block rewards, Proposition 1 allows us to simplify calibration.

Optimal strategy

Denote by q_i^* the optimal selling rate at state i . It is easy to see that if $q_i^* > 0$, then by the HJB equation and the first order condition,

$$P = \frac{\partial V_i}{\partial h} + \frac{\partial C}{\partial q} = \frac{\partial V_i}{\partial h} + \frac{2\lambda P}{h} q_i^*.$$

We then infer that the optimal selling rate q_i^* can be characterized by

$$q_i^* = \frac{h}{2\lambda P} \left(P - \frac{\partial V_i}{\partial h} \right)^+. \quad (18)$$

(18) indicates that (a) there is a positive selling if and only if Bitcoin price is greater than marginal value of holding Bitcoin; and (b) if optimal selling is not zero, then the optimal selling rate should be the one such that the marginal value of holding plus marginal liquidation cost must be equal to Bitcoin price. Therefore, for a given state $i \in \{\mathbb{H}, \mathbb{L}\}$, we can define the i -Selling region and i -Holding region respectively

$$\begin{aligned} i\text{-Selling Region} &= \{(t, x, h) \mid q_i^* > 0\} = \left\{ (t, x, h) \mid P > \frac{\partial V_i(t, x, h)}{\partial h} \right\}, \\ i\text{-Holding Region} &= \{(t, x, h) \mid q_i^* = 0\} = \left\{ (t, x, h) \mid P \leq \frac{\partial V_i(t, x, h)}{\partial h} \right\}. \end{aligned}$$

3.2 Average transaction fee rate

Based on the transaction fees model in Section 2.4, we study the average transaction fee rate. We first consider the average fee rate from an individual miner's prospect. Then we proceed to consider the aggregate average fee rate for a group of miners, which can be examined in terms of market

data.

We first introduce the concept of individual average transaction fee rate. Let Φ^* be the threshold fee rate as given in (11), and $k(\cdot)$ and $K(\cdot)$ be as given in (10). The individual average fee rate r can be defined as

$$r = \frac{K(\Phi^*)}{k(\Phi^*)}. \quad (19)$$

Then we introduce the concept of aggregate average transaction fee rate. In our model, all miners are assumed to be identical, possessing the same successful probability of mining π , which implies that there are $n = 1/\pi$ miners in Bitcoin system. However, these miners may enter the Bitcoin mining market at different time and thus have different inventories. As we have already seen, an individual miner's average fee rate depends on the effective volume of submitted orders L that is relevant to her inventory H (see (12)). As such, different miners may have different average transaction fee rate. We now study the aggregate behavior of transaction fees.

Let Φ_j^* and L_j be respectively the threshold fee rate and the effective volume of submitted orders for miner $j = 1, 2, \dots, n$. The aggregate average fee rate r^A can be expressed as

$$r^A = \frac{\sum_{j=1}^n K(\Phi_j^*)L_j}{\sum_{j=1}^n k(\Phi_j^*)L_j}. \quad (20)$$

The following proposition summarizes the properties of r^A .

Proposition 2. *Assume that there are n miners whose effective volumes of submitted orders are L_j , $j = 1, 2, \dots, n$. Let the aggregate average fee rate r^A be as given in (20).*

(i) *If $L_j < G$ for all $j = 1, \dots, n$, then $r^A = \frac{K(0)}{k(0)}$.*

(ii) *If $L_j \geq G$ for all $j = 1, \dots, n$, then r^A is strictly increasing with the demand X . In particular,*

$$\lim_{X \rightarrow \infty} r^A = \bar{\phi}.$$

Part (i) of Proposition 2 indicates that when the submitted orders are not enough to fill the miner's capacity, i.e. $L \leq G$, the miner will set $\Phi^* = 0$ to take all of the orders, including those orders without transaction fees, and the corresponding average transaction fee rate becomes

constant according to (20). Since a lower demand leads to a lower volume of submitted orders, part (i) suggests that the average transaction fee rate may be flat when the Bitcoin demand is sufficiently low. This is consistent with the market observation that in the period of low transaction demand (e.g. the period of 2013-2016, as shown in Fig. 2), the orders without attaching any fees could be processed and the individual average fee rate remained constant.

Part (ii) of Proposition 2 suggests that when the number of submitted orders exceeds the capacity, both the threshold fee rate and individual average fee rate could increase dramatically. This also coincides with the fact that in the period of high transaction demand (e.g. the year of 2017, as shown in Fig. 2), users need to pay quite a high fee to ensure their orders processed. Part (ii) also asserts that both the threshold fee rate and the average fee rate tend to the upper bound $\bar{\phi}$ assumed in our model. In reality, the existence of the upper bound could be due to the competition from other transaction systems.

With the concept of aggregate average transaction fee rate, we are able to calibrate our model with the observed data of aggregate average fee rate based on the definition of (20).

4 Model Calibration

In this section, we introduce the data used, our calibration method, and calibration results.

4.1 Data and parameter choices

Denote by r_t^A , I_t^A , and H_t^A the aggregate average fee rate, aggregate transaction fees, and aggregate inventory for all miners at time t , respectively. We will use the following available data in Bitcoin system to calibrate our model:

- Monthly Bitcoin price $\{P_t; t = 1, \dots, T_1\}$ from Jan 2013 to Dec 2018 (see Fig. 2);
- Monthly aggregate average fee rate $\{r_t^A; t = 1, \dots, T_1\}$ from Jan 2013 to Dec 2018 (see Fig. 2);

- Monthly aggregate transaction fees $\{I_t^A; t = 1, \dots, T_1\}$ from Jan 2013 to Dec 2018 (see the right panel of Fig. 4);
- Monthly aggregate inventory $\{H_t^A; t = 1, \dots, T_2\}$ from Jan 2013 to Dec 2015 (see Fig. 1);
- Yearly difficulty level in Bitcoin $\{D_y, y = 2013, \dots, 2018\}$ (see the right panel of Fig. 6);
- Daily Mempool size in Bitcoin from Apr, 2016 to Dec, 2018 (see Fig. 7).

Here $T_1 = 72$ and $T_2 = 36$ correspond to Dec 2018 and Dec 2015, respectively. It is worthwhile pointing out that the data of monthly aggregate inventory, which is not directly observable, is obtained by Athey et al., 2016 and available only from Jan 2013 to Dec 2015.

The Bitcoin price P_t is informative to the set of parameters $\Theta_1 = \{\kappa, \nu, \sigma_{\mathbb{H}}, \sigma_{\mathbb{L}}\}$ in (3), while the aggregate average fee rate r_t^A , aggregate transaction fees I_t^A , and aggregate inventory H_t^A are informative to the set of parameters $\Theta_2 = \{\lambda, \theta_{\mathbb{H}}, \theta_{\mathbb{L}}\}$. Calibrating the parameters in Θ_1 and Θ_2 will be elaborated in Section 4.2.

The data of difficulty level D is used to compute miners' probability of successful mining π . Indeed, π can be obtained through formula (5 and the data of D by assuming the computing power for an individual miner $\omega = 5.2TH/s$.⁷ Note that D and π increased dramatically in Bitcoin system from 2013 to 2018 (see Fig. 6), which is against our model assumption that π is invariant. To reconcile the inconsistency,⁸ we stick to the assumption of constant π but update the value of $\pi = \pi_y$ at the beginning of each year in terms of new difficulty level $D = D_y$, $y \in \{2013, \dots, 2018\}$. This also yields the implied number of miners $1/\pi_y$, as showed in the right panel of Fig. 6. For later use, we denote by N_y the number of new miners starting mining business in year $y \in \{2013, \dots, 2018\}$.

Our model assumes two regimes in Bitcoin transaction: high-active regime and low-active regime. In the high-active regime, transaction is very active, implying a large number of orders

⁷Based on Proposition 1, our calibration results are insensitive to the assumption of individual miner's computing power. In fact, from 2013 to 2018, the fraction of transaction fees to block rewards is quite small, indicating that transaction fees may be neglected during this period so that the miners' problem is approximately homogeneous in the computing power and inventory. For simplicity, we choose $\omega = 5.2TH/s$, which implies that there were roughly 10 Bitcoin miners in 2013.

⁸An extension to stochastic π may remove the inconsistency, but it would increase the problem dimension and make calibration even harder.

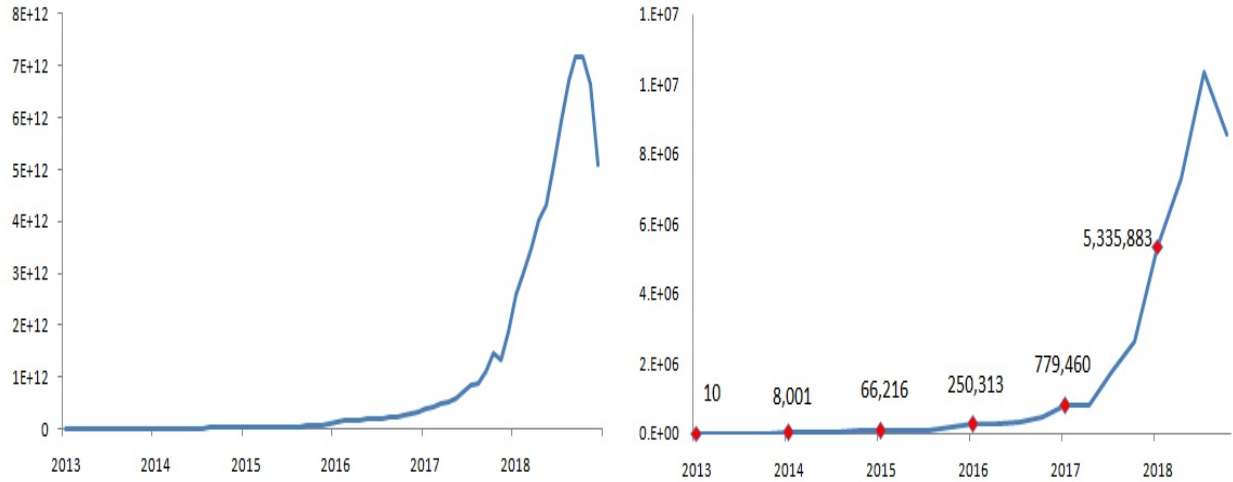


Figure 6: **Difficulty level D and implied number of miners from 2013 to 2018.** Left figure shows difficulty level in Bitcoin system in monthly frequency, and right figure shows implied number of miners in quarterly frequency. Note that we assume each miner has identical computing power $w = 5.22$ TH/s.

submitted by users and high volatility in Bitcoin price. In the low-active regime, transaction of Bitcoin is relatively less active, implying a small number of orders submitted by users and low volatility in Bitcoin price. Using the time series of the Mempool size⁹ as shown in Fig. 7, we can easily identify two regimes. Indeed, one can observe that from the end of 2016 to the beginning of 2018, the moving average of the Mempool size is much higher than that in the rest periods. Since transaction is relatively inactive before 2016, we roughly divide the period of 2013 to 2018 into three parts: low-active regime from 2013 to 2016 Q3; high-active regime from 2016 Q4 to 2017 Q4; low-active regime from 2018 Q1 to 2018 Q4.

Next, we set values for some parameters. We take the discount factor $\beta = 6\%$ that is commonly used in many studies. We normalize the number of Bitcoin to be one, i.e. $\bar{S} = 1$. The corresponding block rewards function becomes $b(t) = \frac{1}{2^{\lfloor t/4 \rfloor + 3}}$ as the Bitcoin rewards halve every four years. For computational simplicity, we assume that the Bitcoin block rewards will be terminated in 2050, after

⁹Mempool is the set of all submitted orders from users in the Bitcoin network. The data of Mempool size is downloaded from www.blockchain.com. It is the aggregate size of transactions (in MB) waiting to be confirmed since 2016. In fact, Biegel (2019) points out that different miner could have slightly different pool of unconfirmed transactions. The data of Mempool size, as a good proxy for the number of submitted orders, is employed to detect the high-active and low-active regimes. A high (low) Mempool size suggests a large (small) number of submitted orders.

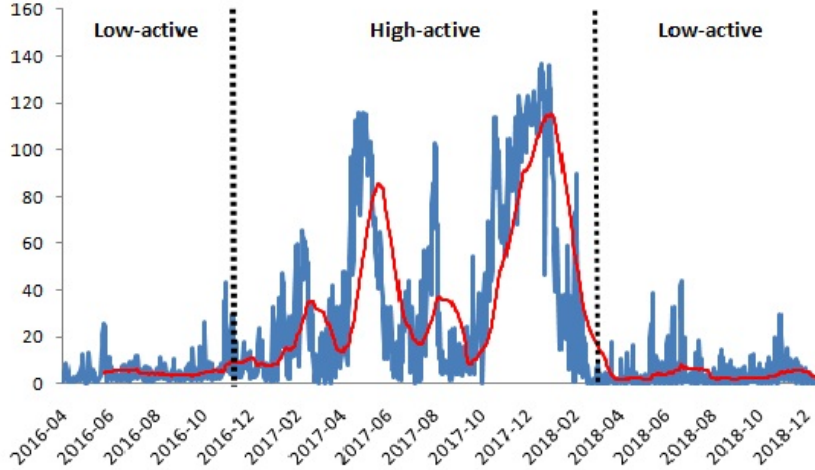


Figure 7: **Two regimes of high-active market and low-active market as suggested by daily Mempool size (in MB) from 2016 to 2018.** Red line is the 60-day moving average.

which block rewards will be tiny and can then be neglected. The capacity of blocks per unit of time satisfies $G = 10$. In fact, the average volume per transaction order since 2016 is around 1.05 Bitcoin. Note that each block can contain around 4000 transaction orders. This suggests that the capacity of number of orders that Bitcoin system allows to process in a year is $4000 \times 6 \times 24 \times 365 = 210,240,000$. Then the normalized capacity per year is $G = 1.05 \times 210,240,000 / 21,000,000 \approx 10$, where 21,000,000 is the total number of Bitcoin. We set $\theta_p = 100$, which implies that one unit change in the demand shock could lead to a change of 100 billion USD in the Bitcoin price. For the parameters in the Beta distribution in (3.4), we set $(a, b) = (0.1, 99.9)$ and $\bar{\phi} = 10\%$. Such parameter values in the Beta distribution function can capture the effect that the average fee rate could increase dozens of times as Bitcoin price changes (see Fig. 2).

We set the transition intensities $\zeta_H = 0.8$ and $\zeta_L = 0.3$, which imply that an average duration of 1.25 years for the high-active regime and of 3.33 years for the low-active regime, respectively. The durations are consistent with what we observe from the dynamics of the Mempool size. Furthermore, we adopt the jump parameters $\lambda_J = 57$ and $Z = 0.9$ estimated by Gronwald (2015), which imply a rough 10% downward jump once a week.

4.2 Calibration for the sets of parameters Θ_1 and Θ_2

We first use Bitcoin price to estimate the parameters $\Theta_1 = \{\kappa, \nu, \sigma_{\text{H}}, \sigma_{\text{L}}\}$ for Gompertz diffusion model of demand shock (i.e. Gutierrez et al., 2004). Based on characterization of high-active and low-active regimes from 2016 to 2018, we estimate the annualized volatility for both high-active and low-active regimes and obtain $\sigma_{\text{H}} = 0.7910$ and $\sigma_{\text{L}} = 0.6225$, which suggest that the demand shock in the high-active regime is more volatile than in the low-active regime. Using the historical Bitcoin price and (2), we obtain the estimated values of the adoption speed of Bitcoin and the log carrying capacity, i.e. $\kappa_1 = \kappa_2 = 1.1742$ and $\nu_1 = \nu_2 = 0.7793$.

It remains to estimate the rest parameters $\Theta_2 = \{\lambda, \theta_{\text{H}}, \theta_{\text{L}}\}$ by matching model outputs with the observed data. We further assume that the miners starting to mine in 2013 have identical inventory and the miners starting to mine in the following years have zero initial holding of Bitcoin. At the beginning of each year, a miner will formulate her objective with the observed difficulty level in the network. Then for the rest months of this year, given the demand level estimated from observed Bitcoin price and the miner's inventory, the miner solves the optimization problem numerically to obtain optimal strategies from month to month. Specifically, with the observed Bitcoin price, we can recover the path of demand shock through time $\{\tilde{X}_t; t = 1, \dots, T_1\}$. For a miner starting to mine in year y , given her inventory $\tilde{H}_{y,t-1}$ at time $t - 1$ and observing demand level \tilde{X}_t at time t , her optimal strategies implied from the model at time t can be computed as follows:

$$\tilde{\Phi}_{y,t} = \Phi_t^*(\tilde{X}_t, \tilde{H}_{y,t-1}), \quad (21)$$

$$\tilde{q}_{i,y,t} = q_{i,t}^*(\tilde{X}_t, \tilde{H}_{y,t-1}). \quad (22)$$

Then at time t the implied transaction fees $\{\tilde{I}_{y,t}, t = 1, \dots, T_1\}$, the implied average fee rate as $\{\tilde{r}_{y,t}, t = 1, \dots, T_1\}$, and the implied inventory $\{\tilde{H}_{y,t}, t = 1, \dots, T_1\}$ for this miner can be computed as follows:

$$\tilde{I}_{y,t} = K(\tilde{\Phi}_{y,t})L(\tilde{H}_{y,t-1}, \tilde{X}_t, i; S_t), \quad (23)$$

$$\tilde{r}_{y,t} = \frac{K(\tilde{\Phi}_{y,t})}{k(\tilde{\Phi}_{y,t})}, \quad (24)$$

$$\tilde{H}_{y,t} = \tilde{H}_{y,t-1} + \pi_y [b(t) + K(\tilde{\Phi}_{y,t})L(\tilde{H}_{y,t-1}, \tilde{X}_t, i; S_t)]\delta t - \tilde{q}_{i,y,t}\delta t, \quad (25)$$

where $\delta t = 1/12$ year as we use monthly data.

By aggregating these results, we obtain aggregate data for all miners as follows, including aggregate transaction fees, aggregate average transaction fee rate, and aggregate inventory:

$$\tilde{I}_t^A = \sum_{y \leq t} N_y \tilde{I}_{y,t} \quad (26)$$

$$\tilde{r}_t^A = \frac{\sum_{y \leq t} N_y K(\tilde{\Phi}_{y,t})L(\tilde{H}_{y,t-1}, \tilde{X}_t, i; S_t)}{\sum_{y \leq t} N_y k(\tilde{\Phi}_{y,t})L(\tilde{H}_{y,t-1}, \tilde{X}_t, i; S_t)} \quad (27)$$

$$\tilde{H}_t^A = \sum_{y \leq t} N_y \tilde{H}_{y,t}. \quad (28)$$

Then we can estimate parameter set $\hat{\Theta}_2$ by minimizing the weighted least square relative error between model output and observed data:

$$\min_{\Theta_2} \frac{1}{T_1} \sum_{t=1}^{T_1} \left\{ \frac{1}{(r_t^A)^2} (r_t^A - \tilde{r}_t^A(\Theta_2))^2 + \frac{1}{(I_t^A)^2} (I_t^A - \tilde{I}_t^A(\Theta_2))^2 \right\} + \frac{1}{T_2} \sum_{t=1}^{T_2} \left\{ \frac{1}{(H_t^A)^2} (H_t^A - \tilde{H}_t^A)^2 \right\}.$$

We use the simulated annealing algorithm (Eglese, 1990), which is good at minimizing a function with many local minima, to find the minimum of our objective function.

4.3 Calibration results

A summary of the estimated parameter values is given in Table 2. As one can see, $\lambda = 0.56$, indicating that there is a significant cost from liquidation for miners. Such liquidation cost could be higher for smaller amount of holding, implying that people may view Bitcoin as a kind of digital art with certain scarcity and miners may prefer to holding at least a small amount of Bitcoins rather than selling all of them. On the other hand, the sensitivity of volume to demand in high-active regime is $\theta_H = 251.3$, which is much higher than $\theta_L = 30.6$ in low-active regime. This is consistent

with the data that the high-active regime has around 10 times of Mempool size than the low-active regime does.

Table 2: Summary of parameters.

Parameters	Symbol	Value
Discount factor	β	6%
Total supply of Bitcoin	\bar{S}	1
Capacity of blocks per unit of time	G	10
Hash rate per miner (TH/s)	ω	5.2
Coefficient in quantity equation (Billion USD per unit)	θ_p	100
Upper bound of fee rate	$\bar{\phi}$	10%
Beta distribution parameters	(a, b)	(0.1, 99.9)
Adoption speed of Bitcoin	κ	1.1742
Log carrying capacity	ν	0.7793
Volatility of demand shock in high-active regime	σ_H	0.7910
Volatility of demand shock in low-active regime	σ_L	0.6225
State transition intensity	(ζ_H, ζ_L)	(0.8, 0.3)
Jump parameters	(λ_J, Z)	(57, 0.9)
parameter in liquidation cost	ψ	0.56
Sensitivity of volume to demand in high-active regime	θ_H	251.3
Sensitivity of volume to demand in low-active regime	θ_L	30.6

In the following we present the calibration results: the model-implied miners' aggregate inventory, the model-implied aggregate average fee rate, and the aggregate annualized transaction fees. We plot the model-implied miners' aggregate inventory in proportion to the supplied Bitcoin in Fig. 8, from which one can see that the model-implied aggregate inventory (the red solid line) perfectly matches the observed one (the black dashed line) from 2013 to 2015. As shown by the green solid line in the figure, the Bitcoin price during this period experienced a drastic increase and a gradual decline. This suggests that our model can explain the puzzle that miners keep reducing their Bitcoin holdings regardless the drastic fluctuation of Bitcoin price. We will show in Section 5 that this may be attributed to the high jump risk in Bitcoin price.

In Fig. 9, the model-implied aggregate average fee rate (in red line) captures two features: 1) In the period of 2014-2016, when demand is low and hence the number of submitted orders is

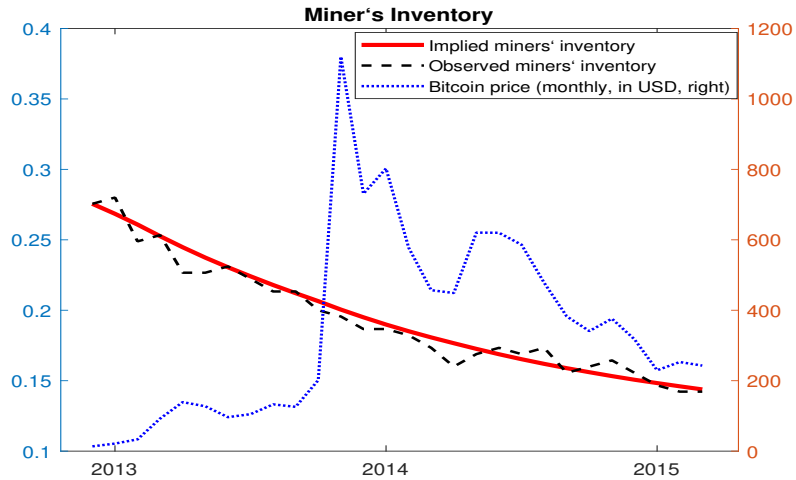


Figure 8: **Implied aggregate inventory from 2013 to 2015 (solid red line)**. Dashed black line shows the observed miners' proportional inventory based on estimation from Athey et al. (2016).

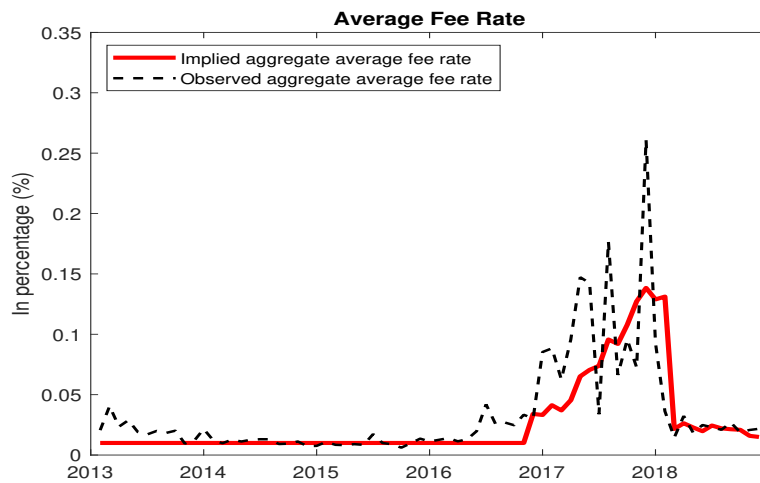


Figure 9: **Implied aggregate average fee rate from 2013 to 2018 (solid red line)**. The dashed black line represents the observed aggregate average fee rate from month to month.

small, miners could accept those orders with zero transaction fee. Therefore, the model-implied average fee rate in this period exhibits little change. 2) As demand goes up from 2016 to 2018, the market switches to a highly active state, which leads the number of submitted order to increase substantially. Then miners prefer to choose those orders with higher threshold fee rate. Therefore the model-implied aggregate average fee rate rises. There are some slight mismatches between implied data and observed data. They may be due to the irrational behaviors in Bitcoin mining

in a short period and imperfect identification of regimes in Bitcoin transactions. Overall, the good match between the model implied results and the observed data suggests that our model provides a reasonable explanation for the dynamics and mechanism of Bitcoin transaction fees.

5 Quantitative Analysis

We conduct an extensive quantitative analysis in this section. First, we numerically characterize the miner’s optimal inventory management strategy. After that, we investigate how demand shock, the miner’s inventory, and system capacity affect the average transaction fee rate.

5.1 The miner’s optimal strategy

Let us examine the miner’s optimal strategy.

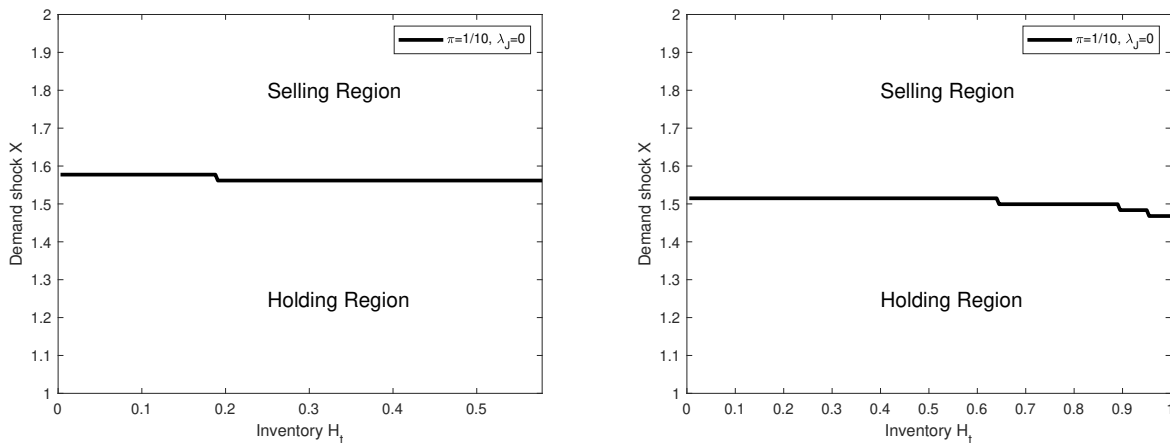


Figure 10: **Selling and holding regions in a high-active market with no jump risk $\lambda_J = 0$ and $\pi = 1/10$.** Parameter values are based on Table 2. Left panel shows the short-run case, i.e. $t = 2014$ and $H_t/S(t) = 0.1$ with $S(t) = 0.5871$. Right panel shows the long-run case when the block rewards is terminated.

Optimal selling barrier, selling region, and holding region. Fig. 10 shows the miner’s inventory management strategies for the short-run case ($t = 2014$, the left panel) and the long-run case (the right panel), where a high-active market with no jump risk ($\lambda_J = 0$) is assumed, the probability of successful mining $\pi = 1/10$, the cumulative supply of Bitcoin at time $t = 2014$ is

$S(t) = 0.5871$ (for short-run case), and other default parameters are reported in Table 2. Numerical results for a low-active market are quite similar and are thus omitted here. Fig. 10 shows that there exists a threshold level of demand, called the (optimal) selling barrier, splitting the solution region into a holding region and a selling region in which the miner holds and sells her inventory, respectively. For a fixed inventory level, the sell region is below the (optimal) selling barrier, because the miner is inclined to reduce her inventory when the demand shock is lower enough, i.e., when the demand shock stays below the selling barrier.

It can be seen from Fig. 10 that the selling barrier decreases as the inventory level increases, because a higher inventory prompts miners to reduce their inventory earlier in order to receive higher transaction fees in the future. The phenomenon of the selling barrier slightly decreasing with inventory for the short run case (the left panel of Fig. 10) is consistent with Proposition 1 that the miner's optimal selling strategy is approximately homogeneous in inventory, as the rewards income significantly prevails over the transaction fee income. As shown by the right panel of Fig. 10 for the long-run case, the selling barrier exhibits a relatively apparent downtrend against inventory, as transaction fees are the unique source of income to miners.

Impact of λ_J , ψ , and π on optimal selling barrier. The jump intensity λ_J , the coefficient of liquidation cost ψ , and the probability of successful mining π have quite different effects on the miner's optimal selling strategy. Fig. 11 plots the selling barrier against jump intensity λ_J for the short-run case ($t = 2014$), where a high-active market is assumed, $\pi = 1/8000$, $S(t) = 0.5871$, $H_t/S(t) = 0.1$, $X = 1$, and other default parameter values are given in Table 2. As jump intensity increases from 0 to 57, the selling barrier decreases from 1.5 to 0. This implies that jump risk could hasten miners' selling of Bitcoin holdings. In particular, for a high jump risk, i.e. $\lambda_J = 57$, miners keep selling their Bitcoin holdings to the market. We then infer that the jump risk is a primary factor determining miners' optimal selling timing. This also explains why miners started to sell their holdings to market at the inception of Bitcoin even when Bitcoin prices were very low.

In addition, liquidation cost could also hasten miner's selling of Bitcoin holdings. As showed in Fig 12, we plot the selling barrier against the coefficient of liquidation cost ψ for the short-run case

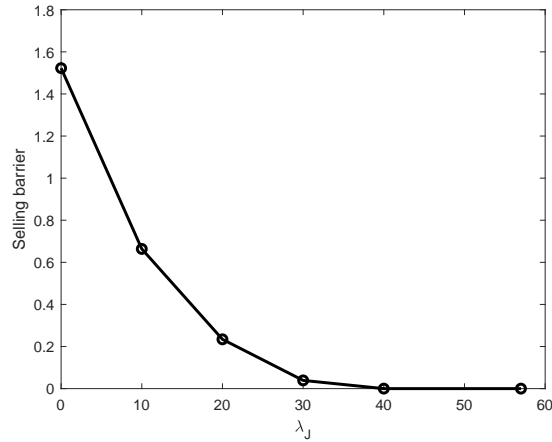


Figure 11: **Impact of jump intensity λ_J on the selling barrier for short-run case in a high-active market with successful mining probability $\pi = 1/8000$.** Parameter values are based on Table 2. We choose $t = 2014$, $H_t/S(t) = 0.1$ with $S(t) = 0.5871$.

($t = 2014$), where a high-active market is assumed, $\pi = 1/8000$, $S(t) = 0.5871$, $H_t/S(t) = 0.1$, $X = 1$, and other default parameter values are given in Table 2. As liquidation cost λ varies from 0.14 to 1.12, the selling barrier decreases from more than 2.1 to lower than 1.3. In fact, facing higher liquidation cost, miners need to optimally choose to sell their holdings earlier to avoid costly liquidation frictions.

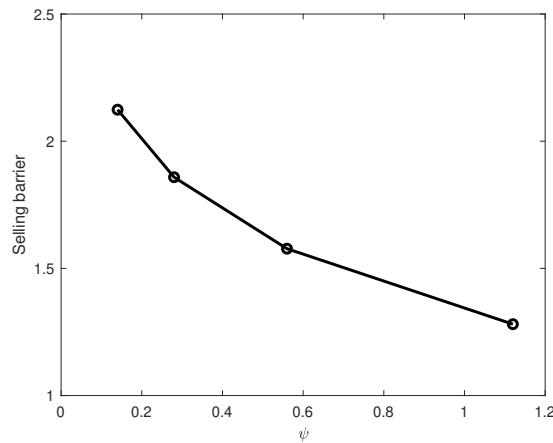


Figure 12: **Impact of coefficient of liquidation cost ψ on the selling barrier for short-run case in a high-active market with successful mining probability $\pi = 1/8000$.** Parameter values are based on Table 2. We choose $t = 2014$, $H_t/S(t) = 0.1$ with $S(t) = 0.5871$.

However, the probability of successful mining π plays a quite different role in miner's optimal

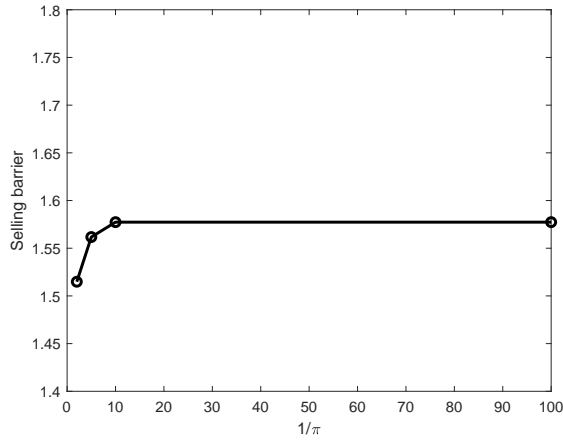


Figure 13: **Impact of probability of successful mining π to the selling barrier for short-run case in a high-active market with no jump risk $\lambda_J = 0$.** Parameter values are based on Table 2. We choose $t = 2014$, $H_t/S(t) = 0.1$ with $S(t) = 0.5871$.

strategy. Fig. 13 plots the optimal selling barrier against $1/\pi$ for the short-run case ($t = 2014$) in a high-active market with no jump risk ($\lambda_J = 0$), where we have fixed $S(t) = 0.5871$, $H_t/S(t) = 0.1$, and a constant demand level $X = 1$. If we increase $1/\pi$ from 2 to 100, the selling barrier increases accordingly. As a lower π is associated with a larger number of miners, this suggests that miners tend to be more reluctant to liquidate their inventories when the number of total miners is raised. The implication behind this phenomenon is that with a larger number of miners, it is more difficult for each miner to get the block rewards and transaction fees, hence, the marginal value of inventory is higher and miners require a higher price (demand level) to sell their holdings. A similar result for the long-run case is reported in the Appendix.

Optimal selling rate. We plot in Fig. 14 the miner's optimal selling rate against demand shock and inventory level for the short-run case ($t = 2014$) in a high-active market with no jump risk $\lambda_J = 0$ and $\pi = 1/8000$. The optimal selling rate is quite smooth, and is non-decreasing with respect to both inventory and demand levels. In Fig. 15, we further examine impacts of π and λ_J on the optimal selling rate for the short-run case ($t = 2014$) in a high-active market, where the miner's holding is fixed at 10% of the total number of Bitcoins, i.e., $H_t/S(t) = 0.1$ with $S(t) = 0.5871$. As shown in Fig. 15, the dashed line, representing the selling rate with a lower $\pi = 1/8000$, is lower than the dotted line that stands for the selling rate with higher $\pi = 1/10$. This implies that a lower

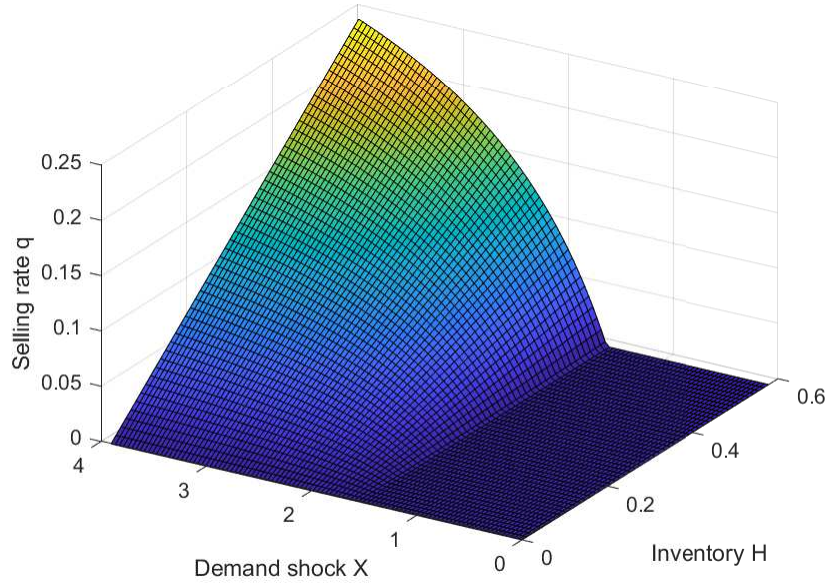


Figure 14: **Optimal selling rate for short-run case in a high-active market with no jump risk $\lambda_J = 0$ and a successful mining probability $\pi = 1/8000$.** We consider $t = 2014$ with $S(t) = 0.5871$, and other default parameter values are as given in Table 2. The miner’s selling rate is non-decreasing with respect to both inventory and demand levels.

successful mining probability makes miners to sell inventory at a lower rate, which is in line with the previous finding that a lower successful mining probability leads miners to defer selling their inventory. On the other hand, the solid line, representing the selling rate with a higher jump risk ($\lambda_J = 57$) but with a lower $\pi = 1/8000$, is much higher than the dotted line with no jump risk ($\lambda_J = 0$) and a higher $\pi = 1/10$, implying that the jump risk is a dominant factor determining the miners’ optimal selling rate and motivates miners to sell fast even at a very low level of Bitcoin demand.

5.2 Average transaction fee rate

Now let us investigate the properties of average transaction fee rate. Fig. 16 shows the miner’s individual average transaction fee rate as a function of demand shock. In a high-active market, the average transaction fee rate in both long-run (black solid line) and short-run (black dashed line) is non-decreasing functions of demand shock. More precisely, the average transaction fee rate

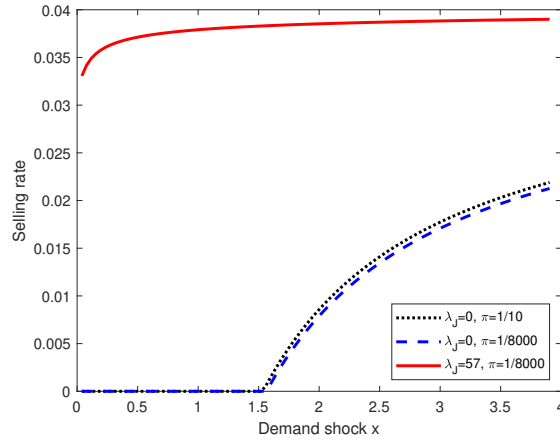


Figure 15: **Optimal selling rate for short-run case in a high-active market.** We consider $t = 2014$ and the miner has Bitcoin holding $H_t/S(t) = 0.1$ with $S(t) = 0.5871$. Other parameter values are based on Table 2.

could be constant for a relatively low demand level, and will increase as demand level increases from a relatively high starting level. Moreover, the average fee rate in long-run is not lower than that in short-run. All these properties are justified by Proposition 2. In fact, cumulative supply of Bitcoin in long-run is higher than that in short-run. Given the same demand shock and the same level of miner's inventory, Bitcoins in market circulation in long-run should be higher than that in short-run. Therefore more transaction orders could be submitted by users in long-run than in short-run, implying a higher average fee rate in long-run. In other words, cumulative supply of Bitcoin by the system could have a positive influence to the transaction fees. Similarly, in a low-active market, long-run and short-run average fee rates (gray solid line and gray dashed line, respectively) are also non-decreasing functions of demand shock. In addition, average fee rate in a low-active market is not higher than that in a high-active market. This is simply because in a high-active market there are a higher level of transaction activities and a larger number of submitted transaction orders comparing to those in a low-active market.

We further examine how the miner's holding affects transaction fees. As shown in Fig. 17, the miner's inventory could have a negative impact to the average fee rate. The left panel of the figure shows that, in a high-active market, the average fee rate when miner's holding is $H = 0.2$, is always not higher than that when miner's holding is $H = 0$. In fact, given the same cumulative supply of

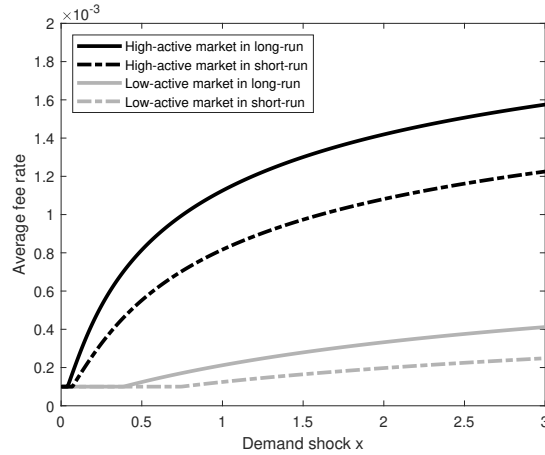


Figure 16: **Individual average fee rate as a non-decreasing function of demand level.** We assume that the miner’s inventory level is zero, i.e. $H = 0$. We assume that the short-run case is at time $t = 2014$.

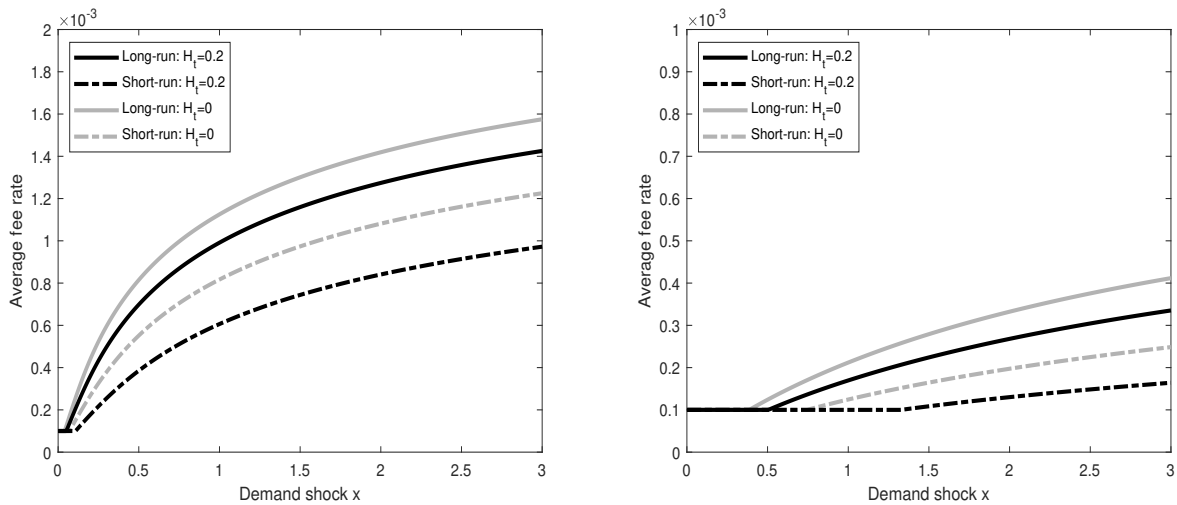


Figure 17: **High inventory level is associated with low individual average fee rate.** We assume that the short-run case is at $t = 2014$. The left panel shows the average fee rate under different inventory levels for both long-run and short-run cases in a high-active market, while the right panel plots those in a low-active market. In both panels, a high inventory level is associated with a low average fee rate.

Bitcoin, the more Bitcoins miners hold, the less Bitcoins are held by users and hence less orders could be submitted by users. Therefore, a higher level of miner's inventory could induce a lower average fee rate for both long-run and short-run cases. The right panel of Fig. 17 shows similar results in a low-active market, and the same conclusion could be achieved.

5.3 Miner's value function

Here we shall examine how the block rewards in declined form and the mechanism of transaction fees affect a miner's value in Bitcoin mining.

Let us fix a demand shock ($X = 1$) and an inventory level ($H = 0$), and investigate how miner's value varies as time increases; see Fig.18. For computational simplicity, we assume that the probability of successful mining $\pi = 1/8000$, and that Bitcoin supply will be terminated in 2050. Miner's value without jump risk in a high-active market and a low-active market are showed in solid black and gray lines, respectively. Meanwhile, the dashed black and gray lines show miner's value with jump risks in both high-active and low-active scenarios. Given that both miner's probability of successful mining and demand of Bitcoin remain constant, we can find that all these value functions are hollow shaped. This is mainly due to the design of block rewards and the mechanism of Bitcoin transaction fees. Note that, other things being equal, miner's value is positively correlated with the total rewards consisting of block rewards and transaction fees. On the one hand, in the early stage of Bitcoin system, miner's income from transaction fees tend to be low as quite few Bitcoins are supplied to market. Therefore, *ceteris paribus*, changes in block rewards play a dominant role in affecting miner's value in this early period. Thus, a decreasing pattern of block rewards, which is predetermined by the system, leads to a decreasing profile of value function with respect to time, *ceteris paribus*. On the other hand, as block rewards decline and more Bitcoins are supplied, the transaction volume increases gradually, which in turn pushes transaction fees up. Then and in contrast, changes in transaction fees play a dominant role in affecting miner's value in this period. Thus, an increasing pattern of transaction fees leads to an increasing profile of value function with respect to time, *ceteris paribus*. Finally, when block rewards are terminated, the miner's value

function is only determined by transaction fees and remains constant.

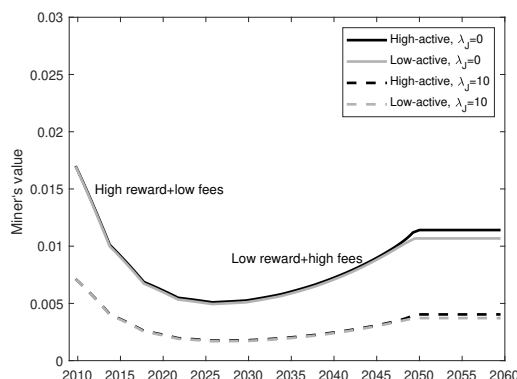


Figure 18: **Impact of block rewards on miner’s value.** Miner’s value is in “U” shape with respect to time given constant demand. Here we assume that the miner’s inventory is zero, i.e. $H = 0$, and the probability of successful mining is $\pi = 1/8000$. The constant demand shock is $X = 1$. For computation simplicity, we assume that Bitcoin supply will be terminated in 2050, and afterwards the miner’s value is the value in long-run.

Furthermore, by comparing the dashed lines (with jump risk) and solid lines (without jump risk) in left plot of Fig. 18, we find that jump risk has a negative effect on miner’s value. Comparing the black lines (high-active) and gray lines (low-active), we also see that miner’s value in a high-active market is higher than that in a low-active market. This is easy to understand since there is a higher transaction demand and more transaction fees could flow to miners in a high-active market.

5.4 Bitcoin After Year 2140

In this subsection, we report simulation results under several scenarios for Bitcoin after year 2140. Given different initial stock price and different level of jump risk, we simulate ten years’ Bitcoin prices since 2140, and plot the corresponding aggregate average transaction fee rate and aggregate inventory for miners. Figures 19-24 report the simulation results under different level of jump risks and of initial Bitcoin prices. For each figure, there are three plots. The first plot shows one simulated Bitcoin price path from 2140 to 2150, the second plot reports the implied aggregate average transaction fee rate based on the simulated Bitcoin price path in the first plot, and the third plot presents the implied aggregate inventory.

The simulation results reveal that miners' aggregate inventory can behave in richer ways subject to different Bitcoin price path, rather than in a decreasing form as shown for the period of 2013 to 2015. They also confirm the mechanism of average transaction fee rate in Bitcoin.

- (i) Even under high jump risk, miner's aggregate inventory could increase for a period and then decrease. Fig.19-21 show the simulation results under a high jump risk, i.e. $\lambda_J = 57$. Fig. 19 is for a low initial price, i.e., 100 USD. Unlike the decreasing form in the period of 2013 to 2015, miners' aggregate inventory increases from 2140 to 2146 and decreases afterwards. This is associated with relative high Bitcoin price from 2140 to 2146 and relative low price from 2147 to 2150. In fact, in the period with relatively high Bitcoin price, miners can collect aggregate transaction fees at a rate higher than their selling rate, and hence their aggregate inventory increases. While in the period with relatively low Bitcoin price, miners face aggregate transaction fees at a rate lower than the selling rate, and therefore the aggregate inventory decreases. Note that in this high jump risk case, miners are always in the selling region. Fig. 20 shows the simulation results with current price level as the initial price, i.e., 10000 USD. At the beginning, aggregate transaction fees is much higher than selling rate, and then the aggregate inventory increases. As long as Bitcoin's price returns to the mean level, the aggregate transaction fees decrease, and then the aggregate inventory decreases. Fig. 21 shows a similar result for the case with high initial Bitcoin price, i.e., 10^6 USD.
- (ii) Miner's aggregate inventory could keep increasing for a long period when facing low jump risk. Figures 22 to 24 report the simulation results under low jump risk, i.e., $\lambda_J = 10$. Note that for the low jump risk case, the miners might be in the holding region and have a low selling rate when Bitcoin demand is low. Our simulation results show that, for the relatively low initial price, the aggregate inventory increases gradually due to the high aggregate transaction fees and low selling rate (see Fig 22 and 23). For the case with high initial price, as shown in Fig. 24, Bitcoin price gradually returns to its mean level. The aggregate inventory increases at the beginning and maintains to be stable level for a long period. This is because miners' transaction fees are quite high in the beginning and maintains to be a stable level for a long

period.

- (iii) High (low) initial Bitcoin price leads to high (low) aggregate average transaction fee rate. As showed in figures 20, 21, 23, and 24, given high level of initial price, the aggregate average fee rate could be higher than 1% for both high and low jump risk scenarios. Meanwhile, for low initial price, aggregate average transaction fee rate could be low or even be a constant, i.e. see Fig. 19 and 22.

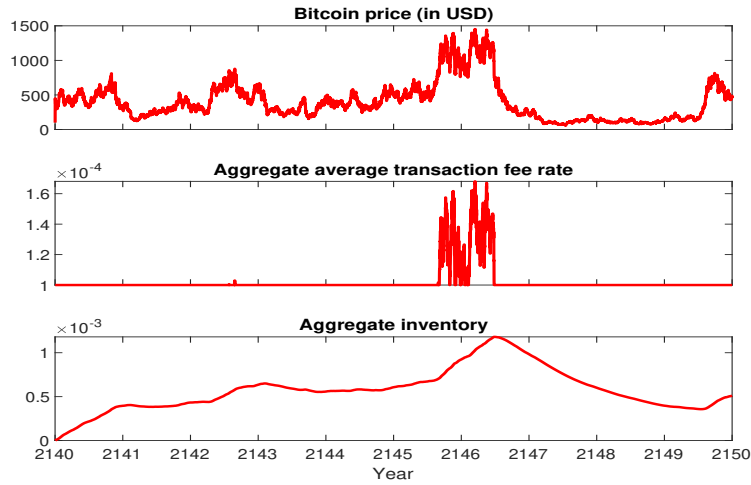


Figure 19: **Simulation results after 2140 with high jump risk and low initial price level.** We assume the initial Bitcoin price is 100 USD, initial aggregate inventory is 0, and $\pi = 1/8000, \lambda_J = 57$. Other parameter values are based on Table 2.

6 Conclusion

We develop a continuous-time dynamic model for Bitcoin mining from the miners’ perspective by borrowing idea of the classic Hotelling model for exhaustible resources. The model is rich enough to incorporate declining Bitcoin rewards and random transaction fees as well as inventory and demand levels, and is able to calibrate to empirical data. We find that high jump risk and transaction fees are major forces driving miners to reduce their inventory even when Bitcoin prices are quite low or very volatile. Our model also explains why the average transaction fee rate stays flat from 2014 to 2016 and increases dramatically in 2017.

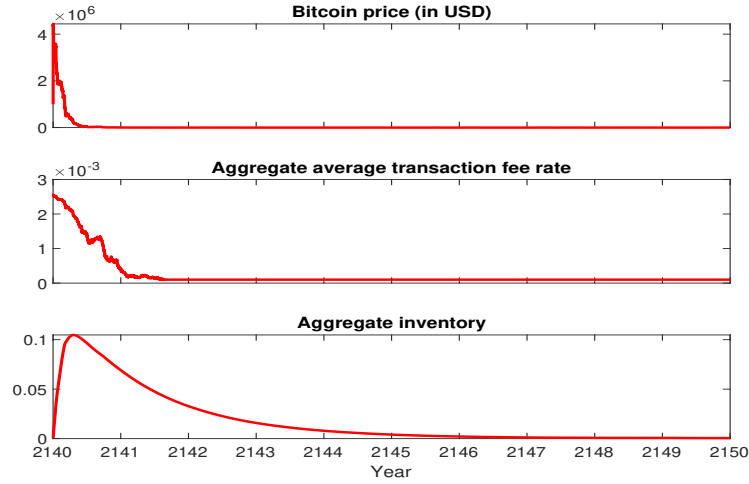


Figure 20: **Simulation results after 2140 with high jump risk and current initial price level.** We assume the initial Bitcoin price is 10000 USD, initial aggregate inventory is 0, and $\pi = 1/8000, \lambda_J = 57$. Other parameter values are based on Table 2.

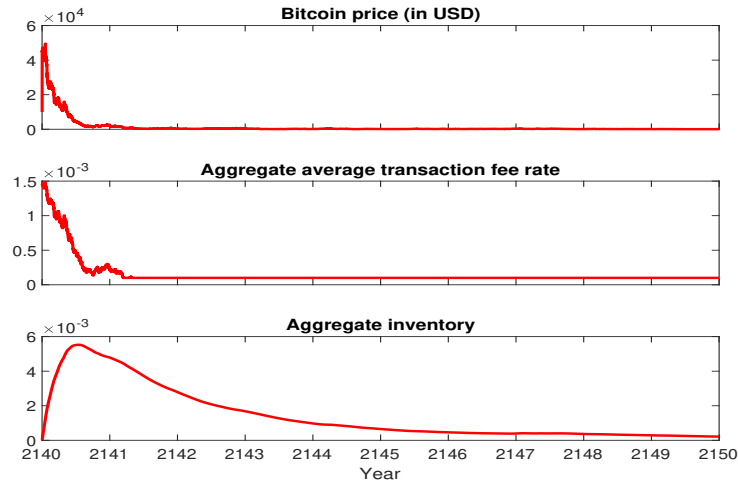


Figure 21: **Simulation results after 2140 with high jump risk and high initial price level.** We assume the initial Bitcoin price is 10^6 USD, initial aggregate inventory is 0, and $\pi = 1/8000, \lambda_J = 57$. Other parameter values are based on Table 2.

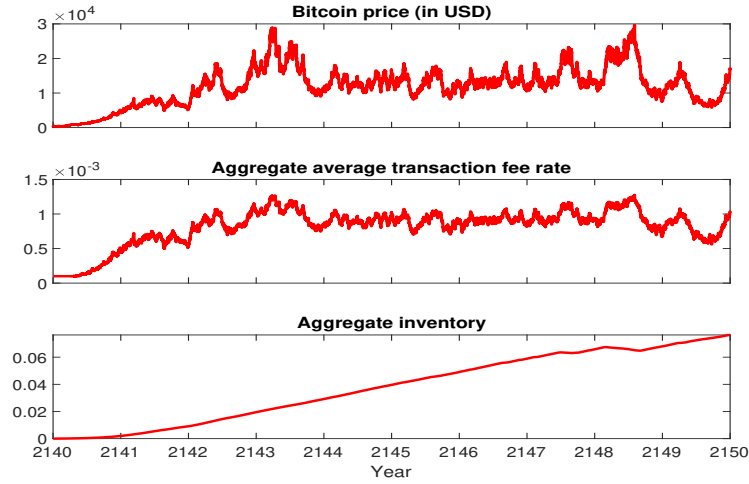


Figure 22: **Simulation results after 2140 with low jump risk and low initial price level.** We assume the initial Bitcoin price is 100 USD, initial aggregate inventory is 0, and $\pi = 1/8000, \lambda_J = 10$. Other parameter values are based on Table 2.

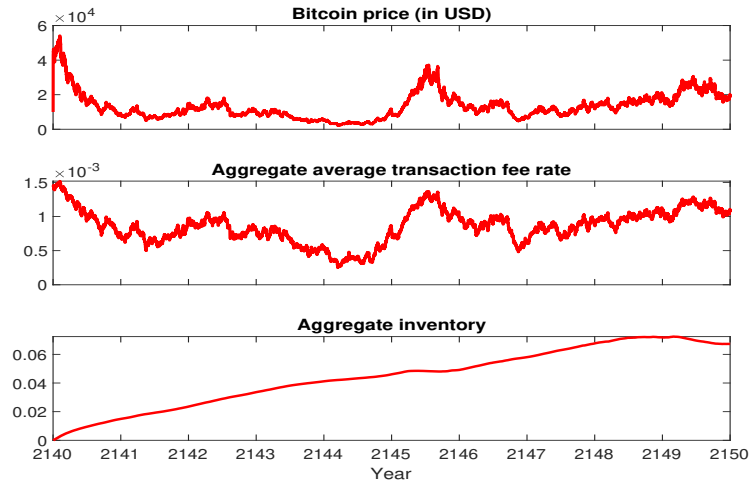


Figure 23: **Simulation results after 2140 with high jump risk and current initial price level.** We assume the initial Bitcoin price is 10000 USD, initial aggregate inventory is 0, and $\pi = 1/8000, \lambda_J = 10$. Other parameter values are based on Table 2.

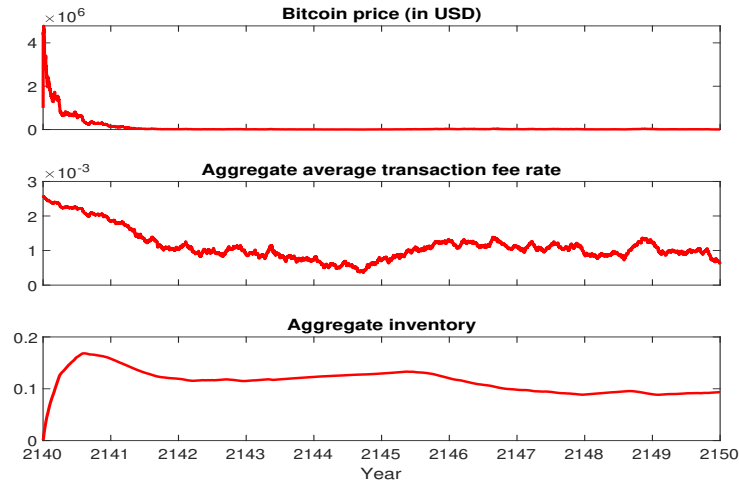


Figure 24: **Simulation results after 2140 with high jump risk and high initial price level.** We assume the initial Bitcoin price is 10^6 USD, initial aggregate inventory is 0, and $\pi = 1/8000, \lambda_J = 10$. Other parameter values are based on Table 2.

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Online Supplement

From Hotelling to Nakamoto: The Economic Meaning of Bitcoin Mining

Appendix

A. Proof of Propositions

A.1. Proof of Proposition 1

Proof. Given an admissible strategy $\{Q_s\}_{s \geq t}$ starting from the state $(t, X_t, H_t; \pi) = (t, x, h; \pi)$, it is easy to see that for any constant $\delta > 0$, $\{\delta Q_s\}_{s \geq t}$ is an admissible strategy starting from the state $(t, X_t, H_t; \pi) = (t, x, \delta h; \delta \pi)$. Hence,

$$\begin{aligned} V(t, x, \delta h; \delta \pi) &\geq \mathbb{E}_t \left[\int_t^\infty e^{-\beta(u-t)} (P_u \delta Q_u - C(\delta Q_u, \delta H_u)) du \right] \\ &= \delta \mathbb{E}_t \left[\int_t^\infty e^{-\beta(u-t)} (P_u Q_u - C(Q_u, H_u)) du \right]. \end{aligned}$$

Taking the supreme over all admissible strategies gives

$$V(t, x, \delta h; \delta \pi) \geq \delta V(t, x, h; \pi).$$

The inverse inequality can be obtained in the same way. □

A.2. Proof of Proposition 2

We first introduce a lemma for individual average transaction fee rate. According to equation (19), apparently both Φ^* and r are non-decreasing as the volume of submitted orders L or the demand X increases. The following lemma reveals that the property of Φ^* and r heavily depends on the

relative magnitude between L and the miner's capacity G .

Lemma 3. *Let the individual average fee rate r be as given in (19).*

(i) *If $L < G$, then $\Phi^* = 0$ and $r = K(0)/k(0)$.*

(ii) *If $L \geq G$, then Φ^* and r are strictly increasing with X (or L). In particular, we have*

$$\lim_{X \rightarrow \infty} \Phi^* = \lim_{X \rightarrow \infty} r = \bar{\phi}.$$

Proof of Lemma 3. By definition, it is easy to check that $K' < 0$ and $k' < 0$. Therefore, the optimal strategy of problem (9) is a corner solution and is given by (11). Assertion (i) thus follows by (11) immediately. For assertion (ii), the first part follows from a direct calculation

$$\frac{\partial r}{\partial X} = \frac{\partial}{\partial X} \left(\frac{K(\Phi^*)}{k(\Phi^*)} \right) = \left(\frac{K(\Phi^*)}{k(\Phi^*)} \right)' \frac{\partial \Phi^*}{\partial X} = \frac{f(\Phi^*) \int_{\Phi^*}^{\bar{\phi}} (u - \Phi^*) f(u) du}{k^2(\Phi^*)} \frac{\partial \Phi^*}{\partial X} > 0.$$

and the fact that L is an increasing function of X . The second part follows by a direct application of the L'Hôpital's rule. \square

Given the results for the individual average transaction fee rate, we now prove the Proposition 2 about aggregate average transaction fee rate.

Proof of Proposition 2. For part (i), we have $L_j \leq G$ for all miners $j = 1, \dots, n$. Based on lemma 3, we have $\Phi_j^* = 0$ and $r_j = \frac{K(0)}{k(0)}$ for $j = 1, \dots, n$. Then we have

$$r^A = \frac{\sum_{j=1}^n K(\Phi_j^*) L_j}{\sum_{j=1}^n k(\Phi_j^*) L_j} = \frac{\sum_{j=1}^n K(0) L_j}{\sum_{j=1}^n k(0) L_j} = \frac{K(0)}{k(0)}.$$

For part (ii), we have $L_j \geq G$ for all miners $j = 1, \dots, n$. We assume there are n miners with inventory satisfying $H_1 \geq H_2 \geq \dots \geq H_n$. Note that L_j for miner j is a decreasing function of H_j . Given the optimal thresh fee rate in (11), we have $L_1 \leq L_2 \leq \dots \leq L_n$ and $\Phi_1^* \leq \Phi_2^* \leq \dots \leq \Phi_n^*$. To examine the monotonicity of aggregate average fee rate in demand, we only need to show that

$$\frac{\partial}{\partial X} (\log \sum_{j=1}^n K(\Phi_j^*) - \log \sum_{j=1}^n k(\Phi_j^*)) > 0.$$

Note that $f(\Phi_j^*) \frac{\partial \Phi_j^*}{\partial X} = \frac{G}{L_j} (X \log(1+X))^{-1}$ and $K'(s) = -f(s)s, k'(s) = -f(s)$. We then obtain

$$\begin{aligned}
& \frac{\partial}{\partial X} (\log \sum_{j=1}^n K(\Phi_j^*) - \log \sum_{j=1}^n k(\Phi_j^*)) \\
&= X^{-2} \left(\sum_{j=1}^n K(\Phi_j^*) \right)^{-1} \left(\sum_{j=1}^n k(\Phi_j^*) \right)^{-1} \left[\sum_{j=1}^n \frac{G}{L_j} \left(\sum_{j=1}^n K(\Phi_j^*) - \sum_{j=1}^n \Phi_j^* k(\Phi_j^*) \right) \right] \\
&= X^{-2} \left(\sum_{j=1}^n K(\Phi_j^*) \right)^{-1} \left(\sum_{j=1}^n k(\Phi_j^*) \right)^{-1} \left[\sum_{j=1}^n \frac{G}{L_j} \sum_{j=1}^n \int_{\Phi_j^*}^{\bar{\phi}} f(s)(s - \Phi_j^*) ds \right] \tag{29}
\end{aligned}$$

We only need to examine the sign in the bracket of (29). Note that $F(\Phi_j^*) = 1 - \frac{G}{L_j}$ and $\frac{G}{L_j} f(s)(s - \Phi_j) = (1 - F(\Phi_j^*)) f(s)(s - \Phi_j)$. We then transform the term in the bracket of (29) to the following form:

$$\begin{aligned}
& \sum_{j=1}^n \frac{G}{L_j} \sum_{l=1}^n \int_{\Phi_l^*}^{\bar{\phi}} f(s)(s - \Phi_j^*) ds = \sum_{j=1}^n \sum_{l=1}^n \int_{\Phi_l^*}^{\bar{\phi}} (1 - F(\Phi_j^*)) f(s)(s - \Phi_j^*) ds \\
&= \sum_{j=1}^n \sum_{l=1}^n (1 - F(\Phi_j^*)) \left[(\bar{\phi} - \Phi_j^*) - (\Phi_l^* - \Phi_j^*) F(\Phi_l^*) - \int_{\Phi_l^*}^{\bar{\phi}} F(s) ds \right] \\
&= \sum_{j=2}^n \sum_{l < j} \left\{ (1 - F(\Phi_j^*)) \left[(\bar{\phi} - \Phi_j^*) - (\Phi_l^* - \Phi_j^*) F(\Phi_l^*) - \int_{\Phi_l^*}^{\bar{\phi}} F(s) ds \right] \right. \\
&\quad \left. + (1 - F(\Phi_l^*)) \left[(\bar{\phi} - \Phi_l^*) - (\Phi_j^* - \Phi_l^*) F(\Phi_j^*) - \int_{\Phi_j^*}^{\bar{\phi}} F(s) ds \right] \right\} \\
&\quad + \sum_{j=1}^n (1 - F(\Phi_j^*)) \left[(\bar{\phi} - \Phi_j^*) - \int_{\Phi_j^*}^{\bar{\phi}} F(s) ds \right] \\
&= \sum_{j=2}^n \sum_{l < j} \left\{ (1 - F(\Phi_l^*)) (\bar{\phi} - \Phi_j^*) + (1 - F(\Phi_l^*)) (\bar{\phi} - \Phi_l^*) - (\Phi_l^* - \Phi_j^*) (F(\Phi_l^*) - F(\Phi_j^*)) \right. \\
&\quad \left. - (1 - F(\Phi_j^*)) \int_{\Phi_l^*}^{\bar{\phi}} F(s) ds - (1 - F(\Phi_l^*)) \int_{\Phi_j^*}^{\bar{\phi}} F(s) ds \right\} \\
&\quad + \sum_{j=1}^n (1 - F(\Phi_j^*)) \left[(\bar{\phi} - \Phi_j^*) - \int_{\Phi_j^*}^{\bar{\phi}} F(s) ds \right] \\
&> \sum_{j=2}^n \sum_{l < j} \left\{ (1 - F(\Phi_j^*)) (\bar{\phi} - \Phi_j^*) + (1 - F(\Phi_l^*)) (\bar{\phi} - \Phi_l^*) - (1 - F(\Phi_j^*)) (\bar{\phi} - \Phi_l^*) \right.
\end{aligned}$$

$$\begin{aligned}
& \left. - (1 - F(\Phi_l^*))(\bar{\phi} - \Phi_j^*) - (\Phi_l^* - \Phi_j^*)(F(\Phi_l^*) - F(\Phi_j^*)) \right\} \\
& = \sum_{j=2}^n \sum_{l < j} \left\{ (1 - F(\Phi_j^*))(\Phi_l^* - \Phi_j^*) + (1 - F(\Phi_l^*))(\Phi_j^* - \Phi_l^*) - (\Phi_l^* - \Phi_j^*)(F(\Phi_l^*) - F(\Phi_j^*)) \right\} \\
& = 0,
\end{aligned}$$

where the inequality is valid because of $\int_{\Phi_j^*}^{\bar{\phi}} F(s)ds \leq (\bar{\phi} - \Phi_j^*)$ for $\forall j \in \{1, \dots, n\}$. This gives the desired result. The limit of r^A with respect to $X \rightarrow \infty$ can be obtained directly from lemma 3.

□

B. Miner's exit decision

Our model can be extended further to consider the miner's optimal exit decision in Bitcoin mining business. For simplicity, we assume no regime-switching in the demand shock of Bitcoin. We aim to examine the miner's optimal exit decision and how the jump risk and other factors affect the decision.

A miner with an exit option has a value function $V(t, x, h)$, in which she chooses optimal stopping time τ_o to maximize her present value of future profit:

$$\sup_{\{Q_u\}_{u \geq t} \in \mathcal{A}_t} \mathbb{E} \left[\int_t^{\tau_o} e^{-\beta(u-t)} (P_u Q_u - C(Q_u, H_u)) du + e^{-\beta(\tau_o-t)} L(X_{\tau_o}, H_{\tau_o}) \right], \quad (30)$$

where $L(X_t, H_t)$ is the value for the miner after exercising the exit option. Here the miner's cost satisfies $C(Q_t, H_t) = C_m + \psi P_t Q_t^2 / H_t$. In our baseline model, constant mining cost C_m does not affect the miner's decision and is thus assumed to be 0. In the model with exit option, the constant mining cost C_m will be included as it is important in making exit decision.¹⁰ Actually, for a miner

¹⁰The mining cost, mainly electricity cost in mining business, can be estimated with average electricity cost and average energy efficiency of mining machines. According to www.eia.gov, the average electricity cost in USA from 2013 to 2018 is 10.3092 cents per Kilowatthour. On the other hand, according to Hayes (2017), the average energy efficiency across the mining network is around 0.4 Watt per GH/s . Hence the mining cost per TH/s is about $C_m = 3.61 \times 10^{-7}$ Billion USD per year.

facing an exit option, she always needs to weigh the return from mining and the cost in mining. She may consider leaving the mining business to avoid high mining cost when the return from mining is low.

Once the miner decides to exit the mining business, she will liquidate her inventory to realize her profit facing an liquidation cost $\psi P_u Q_u^2 / H_u$ and in this case her inventory will only be affected by selling. Therefore, the miner's value function is given by

$$L(X_t, H_t) = \sup_{\{Q_u\}_{u \geq t} \in \mathcal{A}_t} \mathbb{E} \int_t^\infty e^{-\beta(u-t)} \left(P_u Q_u - \psi P_u Q_u^2 / H_u \right) du, \quad (31)$$

where the inventory H_t satisfies $dH_t = -Q_t dt$.

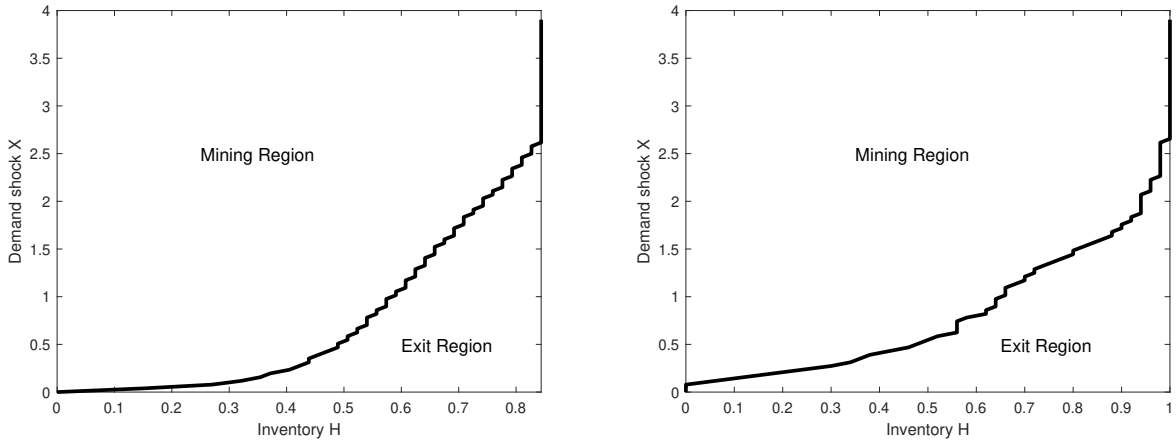


Figure 25: **Exit and mining regions.** We assume $\lambda_J = 0.1, \pi = 1e - 7, C_m = 3.61e - 7$ (Billion USD), and other parameter values are based on Table 2 for the high-activity market. The left panel is for the short-run case, i.e. $t = 2018$ with $S(t) = 0.8438$, while the right panel is for the long-run case in which block rewards are terminated.

We characterize the miner's strategy by two regions: the mining region and the exit region. As showed in Fig. 25, when demand shock is high, as well as the return from mining business, miners are in the mining region. Once the demand shock is too low to cover the expensive operational cost, the miner could be in the exit region and shall optimally leave the mining business. Moreover, a miner with higher Bitcoin holding faces a larger exit region. This is because that a miner with higher Bitcoin holding faces higher liquidation cost and the return from mining could be offset due

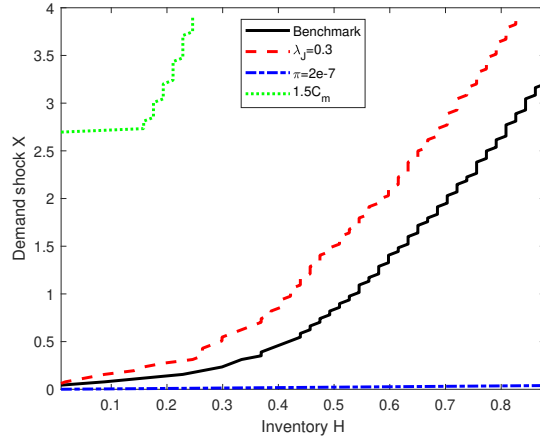


Figure 26: **Impact of λ_J, π, C_m on the exit barriers for short-run case.** We use the exit barrier in Fig. 25 as the benchmark, compared with the barriers with different values of λ_J, π, C_m . Specifically, we consider the three cases: jump risk (λ_J) is tripled, the probability of successful mining π is doubled, and mining cost (C_m) is increased by 50%, respectively. For the short-run case, we have $t = 2018$ with $S(t) = 0.8438$.

to such liquidation frictions. Therefore, a miner with higher holding requires a higher return to stay in mining business. In addition, we plot in Fig. 26 the impact of λ_J, π, C_m on the exit barriers for short-run case and find that increasing in jump risk and operational cost could hasten the miner's exit choice, while increasing in probability of successful mining could delay the miner's exit choice.

C. Additional numerical results

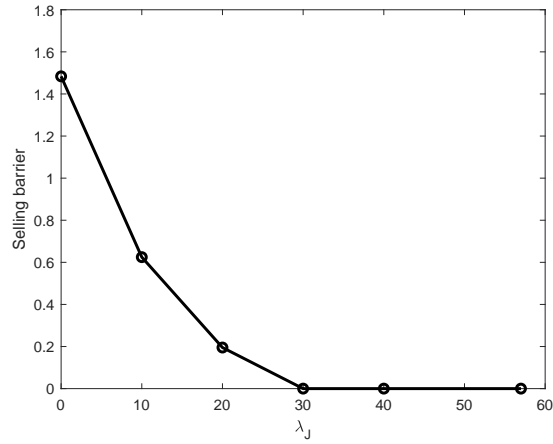


Figure 27: **Impact of jump intensity λ_J to selling barrier for long-run case in High-active market.** Parameter values are based on Table 2. We choose $H_T/S_T = 0.1$ with $S_T = 1$, and $\pi = 1/8000$. Jump intensity λ_J takes values from 0 to 57.

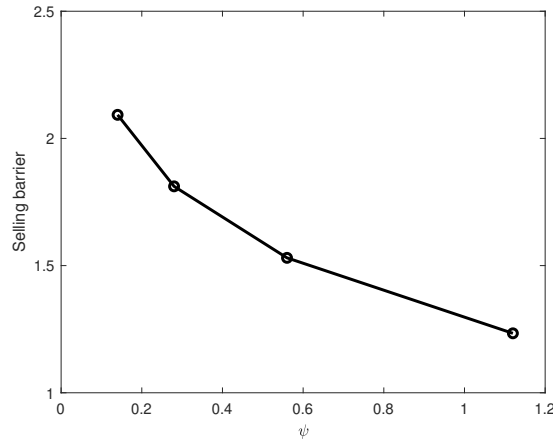


Figure 28: **Impact of coefficient of liquidation cost ψ on the selling barrier for long-run case in a high-active market with successful mining probability $\pi = 1/8000$.** Parameter values are based on Table 2. We choose $H_T/S_T = 0.1$ with $S_T = 1$.

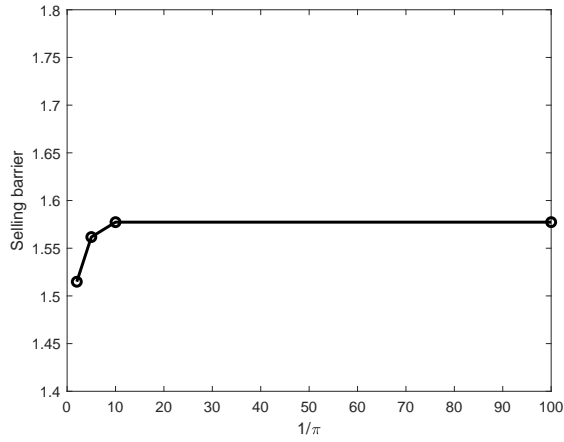


Figure 29: **Impact of successful probability of mining π to selling barrier for long-run case in High-active market.** Parameter values are based on Table 2. We choose $H_T/S_T = 0.1$ with $S_T = 1$. $1/\pi$ takes values from 2 to 100.

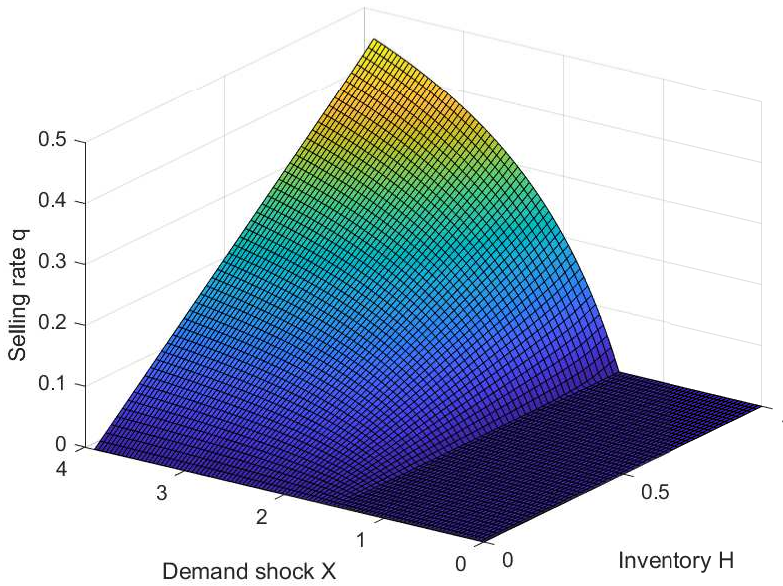


Figure 30: **Optimal selling rate for short-run case in a high-active market with no jump risk $\lambda_J = 0$ and a successful mining probability $\pi = 1/8000$.** Other parameter values are based on Table 2. Miner's selling rate is non-decreasing with respect to both inventory and demand shock.

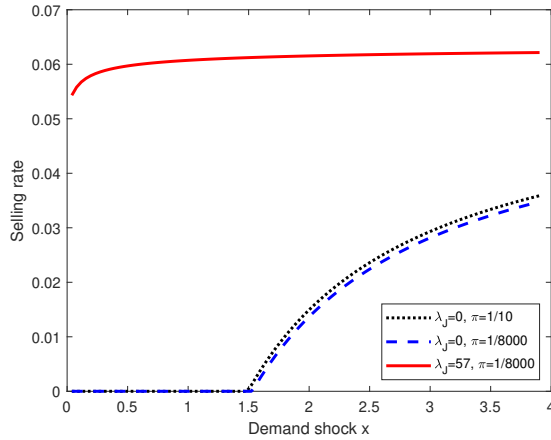


Figure 31: **Optimal selling rate for long-run case in High-active market.** We consider $t = 2014$ and the miner has Bitcoin holding $H_T/S(T) = 0.1$ with $S(T) = 1$. Other parameter values are based on Table 2.

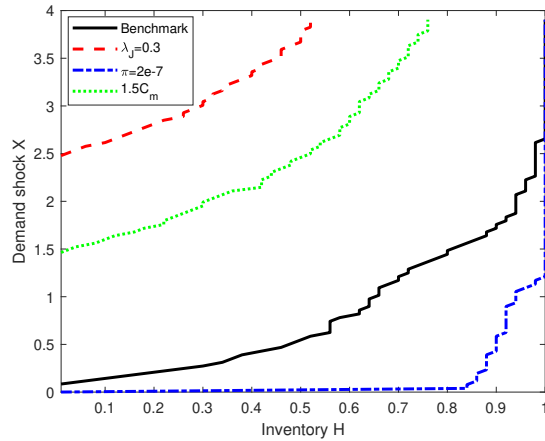


Figure 32: **Impact of λ_J, π, C_m on the exit barriers for long-run case.** We use the exit barrier in Fig. 25 as the benchmark and consider different values in λ_J, π, C_m while remaining the value of other parameters. Specifically, we consider cases including that jump risk is tripled, probability of successful mining is doubled, and mining cost increases 50%.