Endogeneity of Return Parameters and Portfolio Selection: An Analysis on Implied Covariances

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Abstract

The paper presents a method to measure forward looking covariance risk for any two assets even when the explicit market for barter trades does not exist. We argue that the terms of trade in any barter exchanges also follow a martingale process with the condition of no arbitrage. We look at multiple assets with different strike prices. Using a programming approach, we then compute various bivariate risk neutral probabilities for different assets to value all possible pseudo exchange options. This now makes it possible for one to compute implied covariances embedded in the value of any exchange options as in Margrabe [21] even in the absence of the actual exchange option prices. The paper also discusses how these "recoverable" implied return distribution parameters can impact portfolio choice.

Key words: Endogeneity of return parameters; Option implied covariance; Option implied volatility; Forward looking volatility; Forward looking covariance; Forward looking portfolios; Risk neutral probability; Portfolio selection; Capital Asset Pricing Model; Quadratic programming

JEL: G11 and G12

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1. Introduction

Typical portfolio selection models utilize historical mean and variance/covariance returns. Despite the fact that these return parameters themselves are random variables as they are subject to their inherent sampling distribution, we often use them in constructing portfolios without questioning. In addition, we make no specific references as to why the use of certain values of such parameters is particularly correct in portfolio analysis as we often fail to explain how the historical mean and variance/covariance returns are consistent with any market equilibrium.

We have long learned that the market prices signal useful information, and that for example, at least the European option prices give us valuable information about the variance of stock returns with no bias. We say, "no bias," as we believe that the price as a market statistic carries truthful information. Obviously, option embedded volatility does not rely upon particular sample observations and/or statistics. Since the implied volatility is drawn from real-time market prices, they must be forward looking statistic to the extent that the market price looks forward. Why then do we not use these forward looking probability parameters when constructing a portfolio?

The implied volatility does not appear to be unique, however, which may be one of the reasons for not using the implied volatility in portfolio analysis. As is widely known, numerous empirical studies point to the fact that the implied volatility differs for different exercise prices of options, i.e. seeming volatility smiles and frowns; and even for differing maturities, i.e. volatility term structures and/or surfaces. Which volatility number should we use then in our portfolio?

So we often question the validity of our original assumptions about the stochastic process of the underlying securities. Perhaps, the underlying asset prices may not be log-normally distributed, as has been assumed typically in deriving option pricing formulas. Consequently, many researchers conclude that the underlying securities returns themselves may be severely skewed or leptokurtic. Alternatively, we often maintain the log-normality but conjecture that the volatility itself may not be constant or that the underlying returns are mixed primarily with frequent jumps and discontinuities.[3]

What about the *implied* covariance? Unfortunately, the idea of measuring forward looking *implied covariance* poses even yet more difficult problems as neither the market price of exchange barter trades nor the market price for accompanying exchange options are available. Exchange options are those options which allow a barter trade between two non-money assets, e.g. exchanging a share of Microsoft stocks for, let's say, x number of Google shares. In reality, they are not common except perhaps in the case of foreign currencies and stock-for-stock tender offers in a mergers and acquisition trade. The objective of this paper is to show a numerical method to compute the unique value of not only the implied volatilities of a non-money asset but also in particular, the covariances between it and other non-money assets, even when the market for such exchange options does not exist.

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[3] There has been numerous research done in this area and discussing them in detail here would be beyond the scope of this paper.

Section 2, which follows immediately below, presents a possibility that under some circumstances one can actually compute the implied covariances embedded in the market price of various existing options. The basic idea is that if we can somehow find a unique value of *implied risk neutral probabilities* from a multiple number of options with different strike prices but with a given expiration date, we should be able to obtain the unique value of both the implied volatility and the implied covariances even though exchange options may not exist in the marketplace. We do so by looking at the *bivariate* probabilities, and not the marginal density, as we do normally. Obviously, computing the implied covariance is like computing forward looking betas. Section 2 briefly overviews some of these efforts, which have been published in recent years. Some of these works is somewhat *ad hoc* and incomplete, however.

To show how knowing the bivariate probabilities can actually make it possible for us to compute the implicit covariances, we make several "convenient" assumptions: First, we rule out the stock's expected returns as a parameter in determining the value of options to highlight the risk neutral valuation method. Second, we will continue to maintain the log-normality in stock prices and perfect continuity in returns. Consequently, we suppress various empirical evidences of seeming volatility smiles, skews, and the like for different assets at different times.[4] As usual, stock returns are assumed to follow random walks.[5] Thirdly, we assume that the price of an asset can not only be expressed in terms of money, but of the number of units of other non-

[4] For related works and arguments, see Jackwerth and Rubinstein [20] and others.

[5] If stocks follow a *fractal* walk instead, it is known that stocks with a fractal dimension in excess of 1.5 are prone to discontinuous jumps and all jumps would be anti-persistent. See, for example, Rhee [24].

money asset (or assets) as a numéraire for barter. In this circumstance, one can obtain the no arbitrage equilibrium terms of trade in the barter exchange. See Theorem 1. In our paper, this sets the basis for possible pseudo exchange options.

We argue that the value of an exchange option today is the discounted value of the *expected* future payoff of the exchange options at expiration when all underlying assets follow the usual binomial processes. Future payoffs for the options are probability weighted. See equation (5). The probability weights that we use are the risk neutral probabilities for the fact that all options contracts can be replicated and all replicated portfolios are valued without the direct use of the market probabilities; and thus, it is well known that it is possible to compute the implied risk neutral probabilities in this event. We presume that the present value of the expected future payoff of options in equation (5) must be consistent with the value of the option we calculate from Margrabe [21]. We find the Margrabe model directly relevant, as it covers exchange options between any two assets. See Theorem 2. We then finally show the resulting implied covariance in Theorem 3.

Section 3 computes the risk neutral probabilities using a dynamic programming technique Prof. Rubinstein, e.g. [26], first introduced. However, unlike in the case of Rubinstein[6] and also in the traditional derivative pricing literature, we look at the total risk neutral joint probabilities as well as the marginal probability. In other words, we construct joint probability trees to compute implied covariance as well as the implied volatility of the underlying assets. Section 3,

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[6] We find that in some situations, Prof. Rubinstein's model does not yield solution. For a possible reason, please see our discussions in Section 3.

therefore, reviews the Professor Rubinstein's pioneering, quadratic programming method in computing the risk neutral probabilities. We then propose an alternative model to cover the case of more than one asset.

Section 4 integrates the implied variance-covariances into the equilibrium implied expected returns and demonstrates how option premiums can impact optimal portfolio selections. We do so by way of providing an equilibrium comparative statics analysis. Section 5 presents a numerical example. Section 6 summarizes and concludes the paper.

2. Forward Looking Probability Measures for Security Returns in Portfolio Studies

Asset prices are determined endogenously within a system of general equilibrium. We conjecture that the same value of the probability distribution parameters that explains the stock returns must also be able to explain the price of options written on the underlying stocks. It is well known that Capital Asset Pricing Model (CAPM) results from the "general" equilibrium in the market for stocks, but with no specific reference to the existence of derivative markets; and the famous Option Pricing Theory has been developed with no particular reference to the ways in which the stock market reaches equilibrium. As a result, efforts are made to “recover” option embedded implied volatility and covariance returns, which are presumed to look forward. See for example, Jackwerth and Rubinstein [20]. Searching for forward looking covariance is not new, however. We now review some work done to measure forward looking covariances.

2.1 Literature Review

For obvious reasons, many researchers prefer using implied covariances instead of history based correlations in recent portfolio studies. See for example, [6, 7, 9, 13, 29]. A majority of studies have focused on a hybrid time varying correlation structure, e.g. [5], while others simply assumed identical cross correlations, e.g. [23]. French, *et al* [15] computes security betas with the historical covariance relative to option implied volatility. Siegel [27] avoids computing historical covariances but estimates option based security's implied covariance to compute forward looking betas, $\tilde{\beta}_k$, as

$$\tilde{\beta}_k = \frac{\widetilde{Cov}(R_k, R_M)}{\tilde{\sigma}_M^2} = \frac{\tilde{\sigma}_k^2 + \tilde{\sigma}_M^2 - \tilde{\sigma}_e^2}{2\tilde{\sigma}_M^2} \tag{1}$$

To this end, Siegel assumes a theoretical exchange option between a particular stock k and the market index M. Unfortunately, such exchange options between k and M do not exist, which makes it difficult to measure the volatility of the underlying exchange options, $\tilde{\sigma}_e^2$. Christoffersen, *et al* [8] assumes the Single Factor Model and attempts to compute the forwarding looking stock betas as:

$$\beta_i = \left(\frac{skew_i}{skew_M}\right)^{1/3} \left(\frac{var_i}{var_M}\right)^{1/2} \tag{2}$$

The equation is based on the study of Bakshi, *et al* [2], where in particular, the skewness of returns is derived from the value of European call and put options. In their continuous-time Capital Asset Pricing Model (CAPM), Fouque and Kollman [14] simplify the Christoffersen beta with

$$\beta_i = \left(\frac{a^{i,\varepsilon}}{a^{M,\varepsilon}}\right)^{1/3} \left(\frac{b^{i*}}{b^{M*}}\right)^{1/2} \tag{3}$$

by approximating the skewness linearly with respect to the *log-moneyness to maturity* ratio, i.e. $LMMR = \log(K/S)/T$, as

$$\text{Skewness of implied volatilities for } i = b^{i*} + a^{i,\varepsilon}(LMMR); \text{ and}$$

$$\text{Skewness of implied volatilities for } M = b^{M*} + a^{M,\varepsilon}(LMMR)$$

The implied covariance $Cov(R_i, R_j)$ is computed from the implied beta given by equation (3). Unfortunately, we end up "estimating" these parameters using stale *statistical* returns. Admittedly, however, estimators of these parameters themselves are also random variables; and the result can be seen as somewhat *ad hoc* in this sense. In this paper, we offer an alternate solution.

2.2 Basic Idea

In principle, securities' implied covariance must be computable from the market value of "exchange options," just as the implied volatility can be derived from the price of regular stock options. Unfortunately, we run into trouble right away not so much for the underlying theory but for the realistic reason that in most cases, traded "exchange options" are not available in general, as mentioned previously, except foreign currency "exchange options," which may require settling in a certain foreign currency for the delivery of still another foreign currency, e.g.

Euros vs. yens, or yens vs. pounds, etc. Similar exchange options do not exist between stocks. In corporate finance, we often observe merger exchange terms which allows the acquiring firms to offer x number of its own shares for a share of stocks of the acquired, etc., stock for stock tender offers, so to speak. Unfortunately, these are over-the-counter trades, and no such options are traded on organized exchanges. Given these difficulties, we now present some basic economics involved in computing such implied covariance when traded exchange options are not available.

Exchange options are like barter contracts, in which one buys or sells a contract to be able to buy or sell an asset, or the underlying, but to be settled at some strike price yet to be defined in still a different asset, e.g. oil vs. gold, Pepsi vs. vodka, Google vs. Microsoft shares, etc. In this paper, we will deal with a special case where one buys a stock A, with shares of stock B, or vice versa. In other words, there is no cash settlement in this trade.

Suppose now that a trader buys stock B but pays in shares of stock A. Assume that the symbol $S_{B/A}^{t}$ represents the exchange rate of "selling" so many shares of stock A in order to buy one unit of stock B at time t; and $C_{B/A}\left(S_{B/A}^{o}, K_{B/A}\right)$ is the present value of exchange option of buying one unit of stock B at an exchange strike rate of $K_{B/A}$ in stock A at maturity T. The present exchange rate or "price" for this exchange option is $S_{B/A}^{0}$. Then, the following efficient capital market theorem is trivial and should be all familiar to us.

Theorem 1 (No arbitrage)

In an efficient capital market, the price of stock B in terms of the number of shares of stock A is given by $S_{B/A}^{t} = S_{B}^{t}/S_{A}^{t}$ at any time $t \in [0, T]$, where S_{A}^{t} and S_{B}^{t} are dollar price of stocks A and B, respectively.

The proof is quite simple and straightforward, and the assertion is often proven by contradiction. Suppose that the market dictates otherwise that $S_{B/A}^{t} > S_{B}^{t}/S_{A}^{t}$. The trader then can buy a share of stock B, which will cost them S_{B}^{t} dollars. Simultaneously, he or she can exchange the stock B for $S_{B/A}^{t}$ number of shares of stock A for a total dollar gain of $S_{B/A}^{t} \cdot S_{A}^{t}$ by selling back stock A. The trader makes an arbitrage profit, $-S_{B}^{t} + S_{B/A}^{t} \cdot S_{A}^{t} > 0$ with no investment. Similarly, if $S_{B/A}^{t} < S_{B}^{t}/S_{A}^{t}$, he or she can do the opposite by selling the $S_{B/A}^{t}$ units of stock A for one unit of stock B, which will cost the trader $S_{B/A}^{t} \cdot S_{A}^{t}$ dollars. However, stock B is sold for more, i.e. S_{B}^{t} dollars. Either way, the trader makes money. His profit is $-S_{B/A}^{t} \cdot S_{A}^{t} + S_{B}^{t} > 0$, which was originally assumed. This should not be possible in an efficient capital market. Consequently, with no additional wealth committed, the efficient capital market guarantees no arbitrage, i.e. $S_{B/A}^{t} = S_{B}^{t}/S_{A}^{t}$. ■

Consequently, the value of the call exchange option at maturity T with strike price, $K_{B/A}$, can now be stated as

$$max\left(S_{B}^{T}/S_{A}^{T} - K_{B/A}, 0\right). \qquad (4)$$

The fact that S_{A}^{T} and S_{B}^{T} are random variables also suggests that the value of this exchange option is governed by the joint probability distribution. Obviously, if j and k denote all possible *discrete* states of nature, which underlie the stock prices for A and B, and if P_{jk} is the joint probability mass function, i.e. $P_{jk} = Pr\left[S_A^T = S_{A_j}^T, S_B^T = S_{B_k}^T\right]$, and q_A and q_B are yearly dividend yield for stocks A and B, respectively, the value of the exchange option is the present value of the expected maturity value of the option. That is,

$$C_{B/A}\left(S_{B/A}^0, K_{B/A}\right) = e^{(q_B-q_A)T} S_A^0 \sum_{j=1}^{n} \sum_{k=1}^{m} max\left(S_{B_k}^T / S_{A_j}^T - K_{B/A}, 0\right) \cdot P_{jk} \tag{5}$$

We will exclusively deal with discrete probabilities for the ease of illustration.[7] In the next section we will propose ways in which to obtain the joint probability distribution from the current market prices of both stocks and options. *Equation* (5) *is important to us in that it serves as a surrogate to the value of exchange options, which are not otherwise traded in the marketplace.* That is, we treat equation (5) as the true market price of exchange options in an efficient capital market, from which to compute the implied volatility of the exchange rate between two underlying non-money assets. To complete this step, we need a closed form equation for exchange options. For this purpose, we will utilize the pricing formula of the exchange options given in Margrabe [21].

Theorem 2 (Value of Exchange Option)

Assume no arbitrage in the efficient capital market. If the strike rate of an exchange option of an asset A for asset B is given by $K_{B/A}$, and the exchange option is valued as in equation (6)

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[7] In terms of the continuous density,

$$C_{B/A}(S_{B/A}^0, K_{B/A}) = e^{(q_B-q_A)T} S_A^0 \int_0^\infty \int_0^\infty Max\,(S_B^T / S_A^T - K_{B/A}, 0) f(s_A^T, s_B^T) ds_B^T\, ds_A^T$$

where $f(s_A^T, s_B^T)$ is the joint probability density function of S_A^T and S_B^T.

below, then the quantity, $\tilde{\sigma}$, which satisfies the equation is the implied volatility of the terms of trade of A for B, that is $\tilde{\sigma}_{B/A} = \tilde{\sigma}$, provided that $C_{B/A}(S_{B/A}^0, K_{B/A})$ is known. The symbol q is the yearly dividend yield.

$$C_{B/A}(S_{B/A}^0, K_{B/A}) = S_B^0 e^{-q_B T} N(d_1) - S_A^0 K_{B/A} e^{-q_A T} N(d_2) \tag{6}$$

where $d_1 = \dfrac{\ln\left(\dfrac{S_B^0}{S_A^0 K_{B/A}}\right) + \left(q_B - q_A + \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

$N(\cdot)$ is the cumulative standard normal density function.

Proof. Note that $S_{B/A}^0 = S_B^0 / S_A^0$ from Theorem 1. In this case, we also note that an option to exchange a share of stock B for $K_{B/A}$ shares of stock A should be valued in a manner similar to the ways in which the option to exchange two assets are valued in Margrabe, *Ibid.* Therefore if $\tilde{\sigma}$ satisfies the equation (6), it must be the implied volatility of the exchange rate of A for B.

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The next step is to compute implied correlation coefficient and implied covariance with the implied volatility of the exchange rate of two stocks. Letting $R_A \equiv d\ln(S_A^t)$, $R_B \equiv d\ln(S_B^t)$, and $R_{B/A} \equiv d\ln(S_{B/A}^t)$, the implied covariance of two return rates R_A and R_B can be computed by following Theorem 3 below. In this case, we also obtain the implied forward looking beta for any stock.

Theorem 3 (Implied Correlation)

Let $\tilde{\sigma}_A$ and $\tilde{\sigma}_B$ be the implied (as represented by a symbol ~) volatility of R_A and R_B, respectively. Then the implied correlation coefficient and the implied covariance of the return rates are given by equations (7) and (8) as

$$\tilde{\rho}_{A,B} = \frac{\tilde{\sigma}_A^2 + \tilde{\sigma}_B^2 - \tilde{\sigma}_{B/A}^2}{2\tilde{\sigma}_A \tilde{\sigma}_B} \tag{7}$$

$$\widetilde{cov}(R_A, R_B) = \tilde{\rho}_{A,B}\tilde{\sigma}_A\tilde{\sigma}_B = \frac{\tilde{\sigma}_A^2 + \tilde{\sigma}_B^2 - \tilde{\sigma}_{B/A}^2}{2} \tag{8}$$

By replacing the stock B with the market index in Theorem 1, the implied beta for a stock A is given by (9) below where $\tilde{\sigma}_M$ is the implied volatility of market portfolio.

$$\tilde{\beta}_A = \frac{\widetilde{cov}(R_A, R_M)}{\tilde{\sigma}_M^2} \tag{9}$$

Proof. From $S_{B/A}^t = S_B^t / S_A^t$, we obtain $R_{B/A} = R_A - R_B$. Then the variance of $R_{B/A}$ is given by $\sigma_{B/A}^2 = \sigma_A^2 + \sigma_B^2 - 2\rho_{A,B}\sigma_A\sigma_B$ and hence, $\rho_{A,B} = (\sigma_A^2 + \sigma_B^2 - \sigma_{B/A}^2)/(2\sigma_A\sigma_B)$. Now substitute the implied volatilities obtained from Margrabe formula. Then we have $\tilde{\rho}_{A,B}$ by equation (7); $\widetilde{cov}(R_A, R_B)$ by equation (8); and $\tilde{\beta}_A$ by equation (9) as defined, respectively. ■

3. Implied Joint Probability Distribution

We now derive the risk neutral joint probability distribution P_{jk} for the stock prices, S_A^T and S_B^T, at maturity T in order to complete the step in equation (5). Obviously, our goal here is to compute securities correlation and hence, the covariance imbedded in the market price of stocks and options.

To this end, we continue to maintain that in the efficient capital market with no arbitrage, the European call option prices follow the martingale process. See [10,17,25]. The probability as implied in a martingale is in fact the risk neutral measure.[8] In a binomial lattice model, Cox, Ross and Rubinstein (CRR) [11] showed how such risk neutral probabilities, $\hat{P}_j's$, can be computed for a change in the stock price. However, the CRR's analysis was somewhat limited in that the solution may not be unique or even contain an error in their computations especially when there are other options written at different strike prices.

In his 1994 presidential address, Prof. Rubinstein [26] suggested a discrete model using binomial lattice to compute the martingale probability distribution with various call-option prices that has same maturity. He recognizes possible errors, which may arise in the earlier model of CRR, and offers an alternative algorithm to compute the risk neutral probabilities based on a quadratic programming optimization technique. Prof. Rubinstein considered only individual stocks, just as in CRR, and thus, although errors of computations may have been reduced, significant errors appear to have been unhandled as everyone only looked at marginal probabilities when there are other stock options.

We now introduce a method similar to Rubinstein but to allow us to compute the joint

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[8] S_B/S_A is said to be a martingale under the measure $\tilde{P}$ if $S_B^0/S_A^0 = E_{\tilde{P}}[S_B^T/S_A^T]$. The measure $\tilde{P}$ is also referred as the risk neutral measure. If one choose an asset A as the numéraire, $S_B^0 = S_A^0 \cdot E_{\tilde{P}}[S_B^T/S_A^T]$.

probabilities, let alone the stock's marginal probabilities, when the return on one stock is correlated with that on others. While expanding on the Rubinstein's work to the case of more than one stock, our model differs from that of Prof. Rubinstein in that at least at this level, we do not need to assume that the true value of the option lies in between the bid and ask, which we do not believe is totally necessary to implement the Prof. Rubinstein's programming problems. Furthermore, we believe that it is possible that the solution may not even exist, as the feasible solution set in the Prof. Rubinstein's work may be empty. We propose here in this paper an alternative model that always finds a solution.

There have been many other related studies that also explain the seeming volatility smiles, but were focused on a single asset, e.g. [3, 19]. Unlike others, however, we deal with multiple assets. Nonetheless our model is much closer to the traditional CRR model and the Rubinstein's programming model for a single-asset, and thus we now consider the following case of a single asset before we begin our discussions for multi-assets.

3.1 Case of Single Asset

Though our model has been developed for the case of multiple assets, consider the following single stock case for simplicity. We hope that this will bring out all major differences among works of all of us. To this end, we start with the simple one-step binomial lattice model.

3.1.1 One-step binomial lattice model

With only two states considered after the first timestep, we let P_1 be the probability of up move and P_2 be the probability of down move. As usual, we assume in this case that the stock price today, S_0, can rise to S_u or fall to S_d. Similarly for the value of the call written on the stock. See the Figure 1.

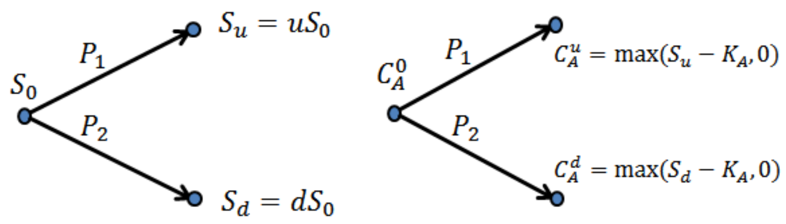


Figure 1. One timestep binomial trees of stock price and call option value

In CRR's model, the probability for the up move is given by

$$P_1 = \frac{a_A - d_A}{u_A - d_A}, \text{where } a_A = e^{r_f T}, u_A = e^{\sigma_A \sqrt{T}}, \text{and } d_A = \frac{1}{u_A}.$$

Here T is the time length of one timestep and σ_A is the volatility of stock A. If we now assume that there is only one option's contract for the stock, then the CRR risk neutral probability $(\hat{P}_1, \hat{P}_2)$ must satisfy the following equations.

$$e^{-r_f T}(S_u P_1 + S_d P_2) = S_0 \qquad (10a)$$

$$e^{-r_f T}\left(C_A^u P_1 + C_A^d P_2\right) = C_A^0 \qquad (10b)$$

The left side of equation $(10a)$ is the present value of the underlying stock, and the right side is the present stock price. Similarly, equation $(10b)$ shows the value of the call option (C_A) equal to the price of the option today. Figure 2 portrays these relations. For example, the dotted line with lesser slope in the figure is for the stock price equation $(10a)$. And the steeper, dashed line in the figure is for the option pricing equation $(10b)$. The solution to the CRR risk neutral probability $(\hat{P}_1, \hat{P}_2)$ is found where two lines intersect. See Figure 2.

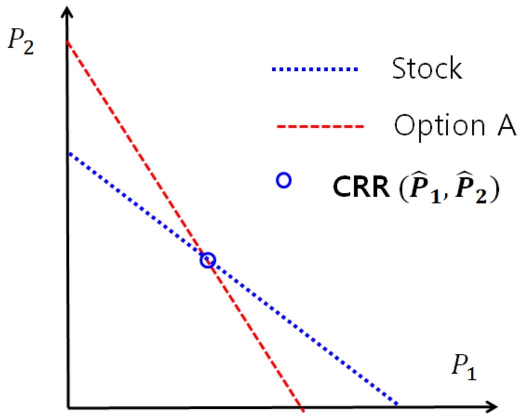


Figure 2. CRR risk neutral probability

Theorem 4 (Implied Risk Neutral Probability)

Suppose that there is a probability pair, $(\hat{P}_1, \hat{P}_2)$ in equation $(10a)$, such that the stock price increases with $\hat{P}_1 = (a_A - d_A)/(u_A - d_A)$ and decreases with $\hat{P}_2 = 1 - \hat{P}_1$, where α_A is a money future value factor, i.e. $a_A = e^{r_f T}$. Then, for any given value of the option, C_A, the pair, $(\hat{P}_1, \hat{P}_2)$, always satisfies equation $(10b)$. That is, the pair $(\hat{P}_1, \hat{P}_2)$ is always implied in C_A.

Proof. The proof is no different than how this theorem was first introduced. Nonetheless, at the risk of redundancy, we provide the following proof in the context of our linear system of equations (10). Suppose that we go long Δ number of shares of stock A for each call written, where $\Delta = (C_A^u - C_A^d)/(S_u - S_d)$. Then, it must follow that

$$S_u\Delta - C_A^u = S_d\Delta - C_A^d$$

The LHS of the equation is the value of the portfolio when the stock price moves up. And the RHS is the value of the portfolio when the stock price moves down. In terms of present value, then,

$$S_0 \ \Delta - C_A^0 = (S_u\Delta - C_A^u)e^{-r_f T} \tag{11}$$

Now, noting that $S_u = S_0 u$, we can rearrange the equation (11) as

$$C_A^0 = S_0\Delta(1 - ue^{-r_f T}) + C_A^u e^{-r_f T} \tag{12}$$

Substituting Δ and rearranging, we now arrive at the following relationship. That is,

$$C_A^0 = e^{-r_f T}\left(P_1 C_A^u + (1 - P_1)C_A^d\right)$$

As it turns out, this is nothing more than equation $(10b)$, and thus, theorem is proved. ■

Introducing the existence of more than one option with different strike prices, Rubinstein now expands the system of linear equations. In the case of two options with different strike prices, and yet, recognizing that the true value of the options should lie in between their respective bid and ask, his risk neutral probability should satisfy the following constraints.

$$S_0^b \leq e^{-r_f T}(S_u P_1 + S_d P_2) \leq S_0^a \qquad (13a)$$

$$C_{A1}^b \leq e^{-r_f T}\left(C_{A1}^u P_1 + C_{A1}^d P_2\right) \leq C_{A1}^a \qquad (13b)$$

$$C_{A2}^b \leq e^{-r_f T}\left(C_{A2}^u P_1 + C_{A2}^d P_2\right) \leq C_{A2}^a \qquad (13c)$$

$$P_1 + P_2 = 1 \qquad (13d)$$

Here, S_0^b and S_0^a are the respective bid and ask prices of stock A. Similarly, C_{A1}^b and C_{A1}^a are the bid and ask for the call options A and $A1$, respectively. Equation $(13a)$ is the present value of the stock. Inequalities in equations $(13b)$ and $(13c)$ are the present values of call option A and $A1$, respectively. If we represent the Rubinstein's model graphically, each constraint must be represented as a band. The solution to the risk neutral probabilities must occur should satisfy the constraints within these bands. Figure 3 shows three pairs of inverse straight lines with each pair of parallel lines representing the pricing relations as computed using either bid or ask prices. In this case, it is clear that there is no unique solution for P_1 and P_2, which satisfies all three pricing relations. However, it must be true that the solution should be found in the shaded common area of intersections of all three pairs of inverse straight lines. Rubinstein chooses the closest point to CRR probability along the line of $P_1 + P_2 = 1$ in the intersection of three bands.

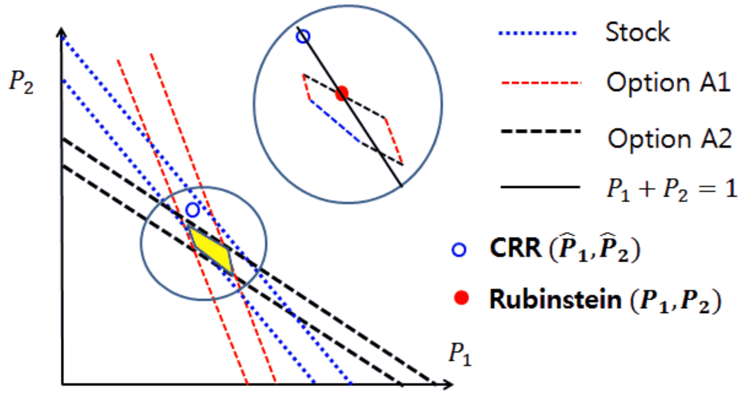


Figure 3. Rubinstein's risk neutral probability in graph.

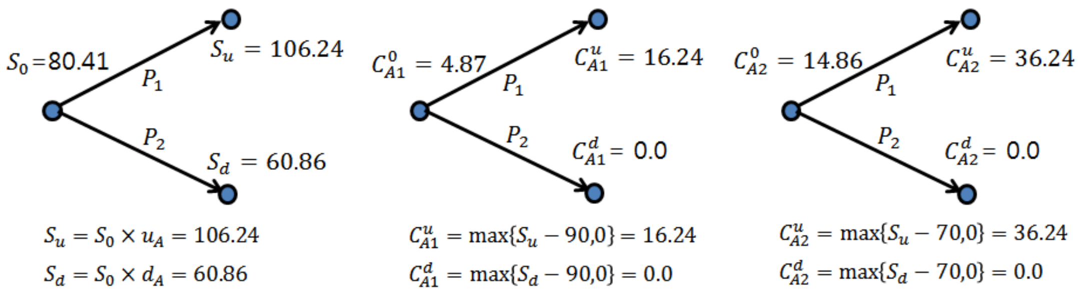
Since the Rubinstein model is suitable for cases with more than one options, which is more realistic, we have been similarly motivated. However, we must mention that his approach is not devoid of its own weakness or criticisms. The serious problem that Rubinstein faces is that if the width between bid and ask prices are narrow, the common intersection of the bands of options may be empty. In this case, Rubinstein offers no solutions for risk neutral probabilities. In addition to the problem just cited, bid and ask prices are awkward to apply, when no solution would exist unless the market microstructure is such that there is enough illiquidity resulting from a sizeable bid-ask spread. The bid and ask spread can quickly disappear at any moment during trading hours. There may be some difficulties to choose exact bids and asks. *Furthermore, one must be reminded of the fact that in equilibrium where all trades occur, the bid would be equal to the ask.* This again means that the Rubinstein model may be suitable only in disequilibrium. The following counter-example shows that it is indeed the case. To prove that this would indeed be the case, we give a counter-example here.

Counter-example. Assume that the bid and ask prices of the stock A and two call options with different strike prices are as in the following table.

Table 1. Bid and ask prices of stock A and two options of counterexample

	Bid price	Ask price	Mean value	Strike price	Time to maturity
Stock A	79.61	81.21	80.41	-	-
Option A1	4.82	4.92	4.87	90	0.885
Option A2	14.71	15.01	14.86	70	0.885

Assume that the stock's annual volatility is $\sigma_A = 29.609\%$. We regard the stock price as the mean value of bid and ask prices. That is, $S_0 = (79.61 + 81.21)/2 = 80.41$, which is in the fourth column of the table. Similarly, the option prices are regarded as the mean values. Therefore $C^0_{A1} = 4.87$ and $C^0_{A2} = 14.86$. The stock's up move, which is then defined as $u_A = e^{\sigma_A\sqrt{T}}$, and the down move, i.e. $d_A = 1/u_A$ are then 1.21769 and 0.82123, respectively.



(a) Stock A (b) Option A1 (c) Option A2

Figure 4. One timestep binomial trees of stock A and two options.

Let the risk free rate be $r_f = 2.14\%$. Then, since the time to maturity is T =0.885, the discount factor is $e^{-r_f T} = 0.9812$. If we substitute the values by corresponding numbers, the inequalities (13a)-(13c) become as follows.

$$79.61 \leq 104.24P_1 + 59.72P_2 \leq 81.21 \qquad (13a)'$$

$$4.82 \leq 15.93P_1 \leq 4.92 \quad \text{or} \quad 0.303 \leq P_1 \leq 0.309 \qquad (13b)'$$

$$14.71 \leq 35.56P_1 \leq 15.01 \quad \text{or} \quad 0.414 \leq P_1 \leq 0.422 \qquad (13c)'$$

Figure 5 represents the inequalities (13a)'-(13c)'. In this case, the common intersection of the bands of options is empty meaning that the risk neutral probabilities as suggested in the Rubinstein's model do not exist.

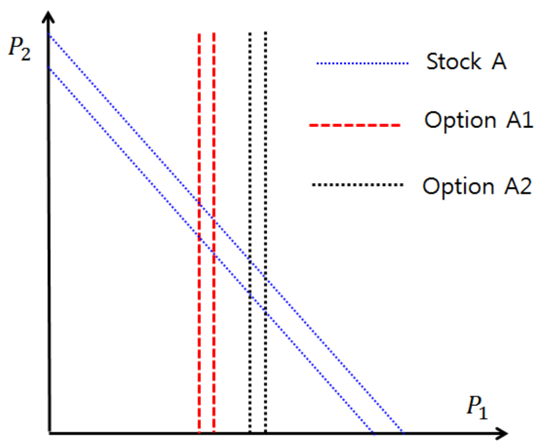


Figure 5. Rubinstein's model fails in this case.

We now present our model, but we use *trade* prices directly instead of bid and ask prices. Remember that the bid and ask cannot be different when trade occurs. However, we will still allow the discrepancy between the value of securities and their prices but in a different way, especially when we consider a single stock and its various options associated with different exercise prices. Obviously, neither Rubinstein's nor ours will make sense if there is only one stock and one option to consider, as the statistical discrepancy would be precisely zero if the market is efficient and functions well. But our objective is to model a system of equations, which will produce the risk neutral probabilities implied in *every* option prices with different strike price. This means that even from the standpoint of each option, its present value and the market price will diverge. One may attribute this discrepancy as "transaction costs" of trading in the spirit of the Rubinstein's model, however. In our case, the transaction costs may also include other execution and/or trading costs. We will simply lump all transaction costs in one place instead of specifying that the value of securities lies somewhere in between the bid and the ask, which is an impractical way to implement the theory empirically. We will henceforth call this discrepancy simply "mathematical" errors. We now present our model.

Our algorithm minimizes the sum of squared errors arising from satisfying the multiple set of linear constraints. It should be noted, however, that these errors may be positive always from the standpoint of high transaction costs, but may as well be negative from the standpoint of search cost of reaching those probabilities, which would be consistent with all options with different prices. In other words, we do not place any constraints to the value (or signs for that matter) of these errors. Our model works whether there is a single option or multiple options. Mathematically, our model has the following constraints:

$$e^{-r_f T}(S_u P_1 + S_d P_2) + \varepsilon_1 = S_0 \qquad (14a)$$

$$e^{-r_f T}\left(C_{A1}^{u} P_1 + C_{A1}^{d} P_2\right) + \varepsilon_{A1} = C_{A1}^{0} \quad (14b)$$

$$e^{-r_f T}\left(C_{A2}^{u} P_1 + C_{A2}^{d} P_2\right) + \varepsilon_{A2} = C_{A2}^{0} \quad (14c)$$

$$P_1 + P_2 = 1 \quad (14d)$$

$$P_1 \geq 0, P_2 \geq 0 \quad (14e)$$

Here, S_0 is the traded price of stock A. And C_{A1}^{0} and C_{A2}^{0} are the traded prices of call options $A1$ and $A2$, respectively, with different exercise prices. $\varepsilon_1, \varepsilon_{A1}$, and ε_{A2} are the errors allowed to occur as a result of *inexact* risk neutral probabilities computed. The solution must occur along the line $P_1 + P_2 = 1$. Note that this optimization problem has always a solution. Figure 6 shows our model graphically. The risk neutral probability by our model is the point in a polygon along the line of $P_1 + P_2 = 1$. Each vertex of the polygon is the intersection point of two lines.

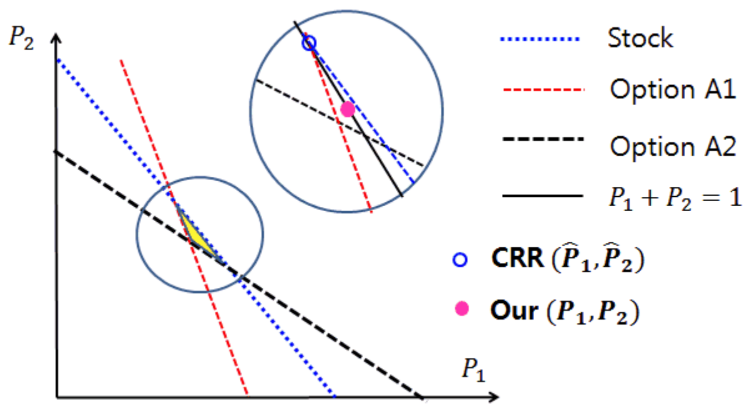


Figure 6. The risk neutral probability in the proposed model

3.1.2 Two-step binomial lattice model

With two time steps, we will again compare the result of the CRR and Rubinstein's model to that of our proposed model.

Assume that the stock A is currently selling for \$80.41 and that there are two call options with different strike prices but both maturing at the same time in almost 11 months, i.e. $t = 0.88493$. One option with the strike price of \$70 is priced at \$14.86, while the other option with the strike price of \$90 is currently at \$4.87. Assume that the stock's annual volatility is 29.609%. The stock does not pay any dividend. The stock's up move, which is then defined as $u_A = e^{\sigma_A \sqrt{\Delta t}}$, and the down move, i.e. $d_A = 1/u_A$ are then 1.21769 and 0.82123, respectively. Then, as is well known, the risk neutral probability for the up move can be computed as, assuming the growth factor $a_A = e^{(r_f - q_A)\Delta t}$,

$$p_A = \frac{a_A - d_A}{u_A - d_A} = 0.4749$$

These market data are as shown in Table 2.

Table 2. The market information of stock A

Stock A	European call option for stock A
Current price: \$80.41	Option A1:
Volatility: σ_A =29.609%	Current price: \$14.86
Risk free rate: r_f =2.14%	Strike price: \$70.0

Time to maturity: 0.88493	Option A2:
Discount factor: 0.9812	Current price: \$4.87
Time step Δt =0.44247 (2 step lattice)	Strike price: \$90.0
Growth factor per step a_A =1.0095	
Up step size u_A =1.21769	Time to maturity for all options:
Down step size d_A =0.82123	0.88493
Probability of up move p_A =0.4749	
Dividend yield: q_A =0.0%	

Now, let us assume that the stock price moves along the two timestep binomial lattice. In the recombining tree lattice, then, the price path of stock A in the first timestep into the future, given the up/down price step size of u, is, as usual, either $S_u = uS_0 = 1.21769 \cdot \$80.61 = \$97.91$ or $S_d = dS_0 = 0.82123 \cdot \$80.61 = \$66.03$. In the second timestep, the expected price at each node is $S_{uu} = u^2 S_0 = 1.21769^2 \cdot \$80.41 = \$119$; $S_{ud} = udS_0 = 1.21769 \cdot 0.82123 \cdot \$80.41 = \$80.41$; and $S_{dd} = d^2 S_0 = 0.82123^2 \cdot \$80.41 = \$54.23$. These results are shown in Figure 7. Value of the two calls at maturity is then $max\{S_{uu} - K, 0\}$, $max\{S_{ud} - K, 0\}$, or $max\{S_{dd} - K, 0\}$. Since the current market price of stock A and its two call option prices are the present value of the future price at timestep 2 *expected* at timestep 1 with no arbitrage, we have

$$S_o \equiv \$80.41 = (0.9812)(119.23\, P_1 + 80.41\, P_2 + 54.23\, P_3)$$

$$C_{A1}^0 \equiv \$14.86 = (0.9812)\{\max(119.23 - 70{,}0)\, P_1 + \max(80.41 - 70{,}0) P_2$$

$$+ \max(54.23 - 70,0)P_3\} = (0.9812)(49.23\, P_1 + 10.41\, P_2)$$

$C^0_{A2} \equiv \$4.87 = (0.9812)(29.23\, P_1)$; and $P_1 + P_2 + P_3 = 1$

In principle, even in the multistep binomial process, the solution to $P_1, P_2,$ and P_3 should exist in this system of linear constraints. In the case of CRR, there always exists the exact solution when there is one stock and one option with one exercise price. Unfortunately, the exact solution to this system usually may not exist if we consider multiple number of options with different exercise prices. So, we ask ourselves, "What probability would approximately satisfy all these linear constraints simultaneously?" Prof. Rubinstein comes to rescue.

Prof. Rubinstein first examines the probability at each node. For example, if the stock price moves up at the timestep 1 and continues to move up even at timestep 2, i.e. node at uu, the risk neutral probability, $\hat{P}_1 = p_A^2 = (0.4749)^2 = 0.2255$. If the price first moves up and then down later, the probability at the node ud or du is $\hat{P}_2 = 2p_A(1 - p_A) = 2(0.4749)(1 - 0.4749) = 0.4987$. The probability at the node dd is $\hat{P}_3 = (1 - p_A)^2 = (1 - 0.4749)^2 = 0.2757$. See Figure 7.

Step 0	Step 1	Step 2	Probability $\hat{P}_j$
80.41	97.9143	119.23	$\hat{P}_1$ =0.2255
	66.035	80.41	$\hat{P}_2$ =0.4987
		54.23	$\hat{P}_3$ =0.2757
			Sum=1.0000

Figure 7. The two-step tree for stock A and the risk neutral probabilities

Since CRR probability considers only one option (usually considers for at-the-money option), it is possible that this approximation may result in a little large "errors" for other options with a different strike price. Prof. Rubinstein argues that the true value of stocks and the corresponding options resides somewhere in between the bid and ask prices (henceforth to be denoted by superscripts b and a in the model, e.g. S_0^b, S_0^a, C_{A1}^b, C_{A1}^a, C_{A2}^b, and C_{A2}^a), and therefore, suggests an alternative quadratic programming model to find P_1, P_2, and P_3 as

$$Minimize \ \ (P_1 - 0.2255)^2 + (P_2 - 0.4987)^2 + (P_3 - 0.2757)^2 \qquad (15a)$$

$$Subject\ to \ \ S_0^b \leq (0.9812)(119.23\, P_1 + 80.41\, P_2 + 54.23\, P_3) \leq S_0^a \qquad (15b)$$

$$C_{A1}^b \leq (0.9812)(49.23\, P_1 + 10.41\, P_2) \leq C_{A1}^a \qquad (15c)$$

$$C_{A2}^b \leq (0.9812)(29.23\, P_1) \leq C_{A1}^a \qquad (15d)$$

$$P_1 + P_2 + P_3 = 1 \qquad (15e)$$

$$P_1 \geq 0,\ P_2 \geq 0,\ P_3 \geq 0 \qquad (15f)$$

In other words, Prof. Rubinstein uses the CRR risk neutral probabilities as a set of initial trial values. And Rubinstein's solution should be close to the CRR probability since the volatility computed by his solution is approximate to the given one of the underlying stock. Therefore, the values in the parenthesis of the objective function $(15a)$, i.e. 0.2255, 0.4987 and 0.2757 are $\hat{P}_1$, $\hat{P}_2$, and $\hat{P}_3$, which CRR would have suggested. So the Rubinstein's solution is the closest point to the CRR probability along the line of $P_1 + P_2 + P_3 = 1$ in the intersection of three bands of $(15b), (15c),$ and $(15d)$. Rubinstein tries to find the risk neutral probabilities that minimize the deviations from the original CRR probabilities. Unfortunately, however, the intersection of the three band constraints may be empty for some values of bid and asks. In addition, using the bid and ask in his research is unrealistic, as the bid and ask changes swiftly by the second. Furthermore, it would be quite difficult to trade using the Rubinstein's solution, when there are multiple trading venues such as not only the organized exchanges but also many other ECNs, even if one uses the National Best Bid and Offer (NBBO) data published under the Consolidated Quotation System.

Our model is more straightforward and yet produces smaller errors. Consider the following quadratic programming problem. We use traded prices instead of bid and ask prices.

$$Minimize \quad \varepsilon_1^2 + \varepsilon_{A1}^2 + \varepsilon_{A2}^2 \tag{16a}$$

$$Subject\ to\ (0.9812)(119.23\ P_1 + 80.41\ P_2 + 54.23\ P_3) + \varepsilon_1 = 80.41 \tag{16b}$$

$$(0.9812)(49.23\ P_1 + 10.41\ P_2) + \varepsilon_{A1} = 14.86 \tag{16c}$$

$$(0.9812)(29.23\ P_1) + \varepsilon_{A2} = 4.87 \tag{16d}$$

$$P_j \leq (1 + \delta) \cdot \hat{P}_j, \qquad j = 1, 2, 3 \tag{16e}$$

$$P_1 + P_2 + P_3 = 1 \tag{16f}$$

$$P_1 \geq 0,\ P_2 \geq 0,\ P_3 \geq 0 \tag{16g}$$

The objective function of this problem minimizes the sum of squared errors along the line $P_1 + P_2 + P_3 = 1$. In this programming problem, it should be noted that unlike in the Rubinstein's model, we also allow a possible error δ between our final risk neutral probabilities and the original CRR probabilities in Eq. (16e). Compare this programming model to our earlier objective function in Eq. (15a). However, the optimization problem set in this way, our programming problem has always a solution. Solution to this programming problem with $\delta = 0.05$ is reported in Table 3 by comparing the traditional CRR risk neutral probability computations, i.e. P_j's and $\hat{P}_j$'s. Clearly, we notice that the expected values of stock and options under our risk neutral probabilities are closer to the market prices than those under CRR risk neutral probabilities with sum of squared errors of 3.83 against the lower 1.62. We could not compare our method with Rubinstein's since it depends on the values of bid prices and ask prices.

Table 3. Comparison of results by two risk neutral probabilities P_j's and $\hat{P}_j$'s

Market information	Real Price	Expected value by CRR prob. $\hat{P}_j$'s	Expected value by our prob. P_j's
Stock A price	80.41	80.41	79.69
Option A1 price	14.86	15.99	15.22
Option A2 price	4.87	6.47	5.86
Sum of squared errors		3.832	1.618
Risk neutral probabilities		$\hat{P}_1 = 0.2255$ $\hat{P}_2 = 0.4987$ $\hat{P}_3 = 0.2257$	$P_1 = 0.2043$ $P_2 = 0.5237$ $P_3 = 0.2720$
Implied volatility of stock A		0.2961	0.2763

Perhaps, the inaccuracy of option values for both in-the-money and out-of-the-money options under the CRR risk neutral probability $\hat{P}_j's$ may have resulted from a possible volatility smile, smirk or even frown. The basic characteristics of the three risk neutral probability measures are summarized below in Table 4.

Table 4. Characteristics of three risk neutral probabilities

	CRR risk neutral probability $\hat{P}_j's$	Rubinstein risk neutral probability $P_j's$	Our risk neutral probability $P_j's$
Finding method	Binomial lattice model	Optimization Model	Optimization Model
Probability distribution (Discrete case)	Binomial distribution	Unrestricted in distribution	Unrestricted in distribution
Limiting distribution (Continuous distribution)	Normal distribution	Unrestricted in distribution	Unrestricted in distribution
Probability distribution of stock price	Log-normal distribution	Unrestricted in distribution	Unrestricted in distribution
Necessary price information	Traded price	Bid price and ask price	Traded price
Valuing option at-the-money	Generally accurate	Generally accurate	Generally accurate
Valuing option in-the-money or out-of-the-money	Inaccurate in some degree	Generally accurate	Generally accurate
Probability (solution) existence	Always exist	May not exist	Always exist
Suitable case	One stock One option	One stock Many options	Two stocks Many options
Performance	-	Better than CRR	Better than CRR
Note	Easy to use	-	Offer the correlation between two stocks

3.2 Case of Two Assets

We now introduce more than one asset, which is necessary to measure the implied correlations explicitly.[9] In particular, we look at the assets' total joint probabilities rather than the marginal probabilities, which only pertain to the single individual asset. In order to extend the model to the case of more than one asset, therefore, we introduce stock B and two call options with different strike prices written on stock B. To expedite, we provide the basic market data for stock B in Table 5 and its accompanying CRR risk neutral probabilities in Figure 8.

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[9] Recently, Bertsimas and Popescu [4] also looked at the case of more than one asset in programming context. A convex optimization technique was used to compute the lower and upper bound of a call-option value, of which decision variable is a continuous probability measure, subjected to the constraints satisfying other call-option prices with different exercise price. They also developed a model with two-stock model of which decision variable is a joint probability measure. The problem to find the continuous probability distribution can be modeled as a semi-definite program (SDP). A typical SDP problem can be stated as a convex programming as follows.

$$Minimize \quad \boldsymbol{C} \cdot \boldsymbol{X}$$

$$Subject\ to \quad \boldsymbol{A_k} \cdot \boldsymbol{X} \leq b_k, k = 1, \dots, n; \text{and}$$

$$\boldsymbol{X} \succcurlyeq 0$$

Here, $\mathbf{C}, \boldsymbol{A_k}$, $\boldsymbol{X}$ are $n \times n$ matrices, and the matrix inner product, $\boldsymbol{C} \cdot \boldsymbol{X} = \sum_{i,j=1}^{n} C_{ij} X_{ij}$. The last constraint $\boldsymbol{X} \succcurlyeq 0$ requires symmetric and positive semi-definite matrix $\boldsymbol{X}$. Some efficient algorithms for SDP are found in [1, 22, 28].

Table 5. The market information of stock B

Stock B	European call option for stock B
Current price: $555.48	Option B1:
Volatility: σ_B =22.088%	Current price: $60.30
Risk free rate: r_f =2.14%	Strike price: $530.0
Time to maturity: 0.88493	Option B2:
Discount factor: 0.9812	Current price: $26.10
Time step Δt =0.44247 (2 step lattice)	Strike price: $600.0
Growth factor per step a_B =1.0095	Time to maturity for all options:
Up step size u_B =1.15827	0.88493
Down step size d_B =0.86336	
Probability of up move p_B =0.49559	
Dividend yield: q_B =0.0%	

Step 0	Step 1	Step 2	Probability $\hat{P}_k$
555.48	643.396	745.23	$\hat{P}_1$ =0.24561
	479.577	555.48	$\hat{P}_2$ =0.49996
		414.05	$\hat{P}_3$ =0.25443
			Sum=1.0000

Figure 8. The two-step tree for stock B and the CRR risk neutral probabilities

To show the performance of our risk neutral joint probability P_{jk}, define $\hat{P}_{jk}$ as the product $\hat{P}_{j.}$ (the probability of being at node j for stock A) and $\hat{P}_{\cdot k}$ (the probability of being at node k for stock B). That is,

$$\hat{P}_{jk} = \hat{P}_{j.} \times \hat{P}_{\cdot k} = \begin{pmatrix} 0.055 & 0.113 & 0.057 \\ 0.122 & 0.249 & 0.127 \\ 0.068 & 0.138 & 0.070 \end{pmatrix}$$

Then $\hat{P}_{jk}$ is the joint probability when stock A and stock B are independent. We will compare our joint probability P_{jk} with CRR joint probability $\hat{P}_{jk}$. Now we want to find our risk neutral probability P_{jk} satisfying the following optimization problem. This optimization problem is equivalent to the problem (16) except choosing the stock A as the numeraire instead of riskless bond.

$$Minimize \quad \sum_{i=1}^{2} (\varepsilon_i^2 + \varepsilon_{Ai}^2 + \varepsilon_{Bi}^2) \tag{17a}$$

$$Subject\ to$$

$$(80.41)(P_{1.} + P_{2.} + P_{3.}) + \varepsilon_1 = 80.41 \tag{17b}$$

$$(80.41)\left(\frac{745.23}{119.23}P_{11} + \frac{745.23}{80.41}P_{21} + \frac{745.23}{54.23}P_{31} + \frac{555.48}{119.23}P_{12} + \frac{555.48}{80.41}P_{22} + \frac{555.48}{54.23}P_{32} + \frac{414.05}{119.23}P_{13} + \frac{414.05}{80.41}P_{23} + \frac{414.05}{54.23}P_{33}\right) + \varepsilon_2 = 555.48 \tag{17c}$$

$$(80.41)\left(\frac{49.23}{119.23}P_{1.} + \frac{10.41}{80.41}P_{2.}\right) + \varepsilon_{A1} = 14.86 \tag{17d}$$

$$(80.41)\left(\frac{29.23}{119.23}\,P_{1.}\right) + \varepsilon_{A2} = 4.87 \tag{17e}$$

$$(80.41)\left(\frac{215.23}{119.23}P_{11} + \frac{215.23}{80.41}P_{21} + \frac{215.23}{54.23}P_{31}\right.$$
$$\left. + \frac{25.48}{119.23}P_{12} + \frac{25.48}{80.41}P_{22} + \frac{25.48}{54.23}P_{32}\right) + \varepsilon_{B1} = 60.30 \tag{17f}$$

$$(80.41)\left(\frac{145.23}{119.23}P_{11} + \frac{145.23}{80.41}P_{21} + \frac{145.23}{54.23}P_{31}\right) + \varepsilon_{B2} = 26.10 \tag{17g}$$

$$P_{j.} = P_{j1} + P_{j2} + P_{j3} \le (1+\delta)\hat{P}_{j.}\,, j = 1, 2, 3 \tag{17h}$$

$$P_{.k} = P_{1k} + P_{2k} + P_{3k} \le (1+\delta)\hat{P}_{.k}\,, k = 1, 2, 3 \tag{17i}$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3} P_{jk} = 1 \tag{17j}$$

$$P_{jk} \ge 0, j = 1, 2, 3;\ k = 1, 2, 3 \tag{17k}$$

In the optimization problem (17), the first two constraints are for values of stock A and B with the stock A as the numeraire. The next four constraints are for option values of different strike prices of stock A and stock B. The values are evaluated by stock A price as the numéraire. Inequalities (17h) and (17i) are introduced for volatilities of stock A and B. The comparison of results of two risk neutral probabilities P_{jk}'s and $\hat{P}_{jk}$'s is given by Table 6 with $\delta = 0.05$. Note that the expected values of stock and options under our risk neutral probability are closer to the market prices than those under CRR risk neutral probability. In this two-stock case, inaccuracy under CRR risk neutral probability also arises from its independent assumption. The closer the relation of two stocks is, the bigger the inaccuracy is. Though this example does not include any related constraints of two stocks, but such constraints should be required in an extended model (to be introduced later).

Table 6. Comparison of results by two risk neutral probabilities P_{jk}'s and $\hat{P}_{jk}$'s

Market information		Real price	Expected value by CRR prob. $\hat{P}_{jk}$'s	Expected value by our prob. P_{jk}'s
Stock A price		80.41	80.41	80.41
Option A1 price		14.86	12.68	13.31
Option A2 price		4.87	4.45	4.67
Stock B price		555.48	599.88	559.93
Option B1 price		60.30	69.52	53.13
Option B2 price		26.10	37.80	27.74
Sum of squares of errors			2198.0	76.40
Risk neutral probabilities			$\begin{pmatrix} 0.055 & 0.113 & 0.057 \\ 0.122 & 0.249 & 0.127 \\ 0.068 & 0.138 & 0.070 \end{pmatrix}$	$\begin{pmatrix} 0.205 & 0.011 & 0.021 \\ 0.053 & 0.464 & 0.007 \\ 0.000 & 0.000 & 0.239 \end{pmatrix}$
Implied volatility	Stock A		0.2961	0.2820
	Stock B		0.2209	0.2170

We now formalize the model to obtain $P_{jk} = Pr\,\{S_A^T = S_{A_j}^T, S_B^T = S_{B_k}^T\}$ used in equation (5) to compute the value of the exchange option. We use two dimensional binomial lattice to obtain the joint probability distribution where the exchange ratio of two stocks is martingale.

Define:

q_A = the yearly dividend rate of stock A

q_B = the yearly dividend rate of stock *B*

j = the node index of stock *A*, $j = 1, \cdots, n$

k = the node index of stock *B*, $k = 1, \cdots, n$

A_j = the price at the j-th node of stock *A*, $j = 1, \cdots, n$

B_k = the price at the k-th node of stock *B*, $k = 1, \cdots, n$

T = the time to maturity (We assumed that every option has the same maturity time)

$P_{jk} = Pr\{S_A^T = A_j, S_B^T = B_k\}$, decision variables, $j = 1, \cdots, n$, $k = 1, \cdots, n$

$P_{j.}, P_{.k}$ = marginal probabilities, $P_{j.} = \sum_{k=1}^{n} P_{jk}$ $(j = 1, \cdots, n)$ and $P_{.k} = \sum_{j=1}^{n} P_{jk}$ $(k = 1, \cdots, n)$

$\hat{P}_{jk} = Pr\{S_A^T = A_j, S_B^T = B_k\}$, CRR probability when S_A^T and S_B^T are independent, $j, k = 1, \cdots, n$

$\hat{P}_{j.}, \hat{P}_{.k}$ = marginal probabilities, $\hat{P}_{j.} = \sum_{k=1}^{n} \hat{P}_{jk}$ $(j = 1, \cdots, n)$ and $\hat{P}_{.k} = \sum_{j=1}^{n} \hat{P}_{jk}$ $(k = 1, \cdots, n)$

$S_{A_j}^T = A_j$: the *j*-th node price of stock *A* at maturity *T*

$S_{B_k}^T = B_k$: the *k*-th node price of stock *B* at maturity *T*

$K_A^{i_1}$= the i_1-th exercise price of European call-option of stock *A*, $i_1 = 1, \cdots, n_1$

$K_B^{i_2}$= the i_2-th exercise price of European call-option of stock *B*, $i_2 = 1, \cdots, n_2$

$C_A^{i_1}$ = the present price of European call-option of stock A with exercise price $K_A^{i_1}$, $i_1 = 1, \cdots, n_1$

$C_B^{i_2}$ = the present price of European call-option of stock B with exercise price $K_B^{i_2}$, $i_2 = 1, \cdots, n_2$

$\varepsilon_1, \varepsilon_2, \varepsilon_A^{i_1}, \varepsilon_B^{i_2}$ = the additional variables introduced to minimize the differences in constraints

δ = the parameters that show the allowable tolerance in marginal probability related constraints

Recall that the symbol $S_{B/A}^{t}$ represents the exchange rate of "selling" so many shares of stock A in order to buy one unit of stock B at time t; and the present exchange rate or "price" for this exchange option is $S_{B/A}^{0}$. Now we suggest the following quadratic programming problem, **QP,** to derive the risk neutral joint probability distribution.

$$Minimize \quad \varepsilon_1^2 + \varepsilon_2^2 + \sum_{i=1}^{n_A} \left(\varepsilon_A^{i_1}\right)^2 + \sum_{i=1}^{n_B} \left(\varepsilon_B^{i_2}\right)^2 \tag{18a}$$

$$Subject\ to \quad S_A^0 \sum_{j=1}^{n} P_{j.} + \varepsilon_1 = S_A^0 \tag{18b}$$

$$S_A^0 \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{S_{B_k}^T}{S_{A_j}^T} P_{.k} + \varepsilon_2 = S_B^0 \tag{18c}$$

$$S_A^0\, e^{q_A T} \sum_{j=1}^{n} \frac{\max\,(S_{A_j}^T - K_A^{i_1}, 0)}{S_{A_j}^T}\, P_{j.} + \varepsilon_A^{i_1} = C_A^{i_1}, i_1 = 1, \cdots, n_1 \tag{18d}$$

$$S_A^0\, e^{q_B T} \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\max\left(S_{B_k}^T - K_B^{i_2}, 0\right)}{S_{A_j}^T}\, P_{jk} + \varepsilon_B^{i_2} = C_B^{i_2}, i_2 = 1, \cdots, n_2 \tag{18e}$$

$$P_{j.} \leq (1+\delta)\hat{P}_{j.}\,, j = 1, \cdots, n \tag{18f}$$

$$P_{.k} \leq (1+\delta)\hat{P}_{.k}\,, k = 1, \cdots, n \tag{18g}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} P_{jk} = 1 \tag{18h}$$

$$P_{jk} \geq 0, j = 1, \cdots, n;\ k = 1, \cdots, n \tag{18i}$$

The objective function is sum of squared errors, which is to be minimized. The first two constraints $(18b)$ and $(18c)$ require that the present value of each stock should be the market price with the stock A as the numeraire. Equation $(18d)$ also states that the present value of call-option of stock A with exercise price $K_A^{i_1}$ should be the market price of the option. The same is true with the value of call-options of stock B, it is shown as equation $(18e)$. In equations (18d) and (18e), call option values are evaluated by stock A price as the numeraire. Inequalities (18f) and (18g) are for volatilities of stock A and stock B, and equations (18h) and $(18i)$ are required for P_{jk} to be a probability measure.

As the solution of above QP, the risk neutral probabilities P_{jk}'s are necessary to obtain the value of exchange option from the equation (5) in the previous section. The more accurate risk neutral probabilities of QP will be given in Section 5. Here we show how to calculate the value of exchange option with simple example.

Simple Example (Value of exchange option and the implied covariance)

Consider the above QP with two timesteps $(n = 3)$. Now let the joint probability

$\begin{pmatrix} 0.205 & 0.011 & 0.021 \\ 0.053 & 0.464 & 0.007 \\ 0.000 & 0.000 & 0.239 \end{pmatrix}$ of Table 6 be the solution of QP with constraints $(18b - i)$. Even though the probability is the solution of two timestep case, we will use it only with the purpose of showing the process of calculation. In this example $S_A^0 = \$80.41$ and $S_B^0 = \$555.48$. And thus the present exchange ratio $S_{B/A}^0 = 6.908$. Let K_A=70 and $K_B = 530$. Then $K_{B/A} = 530/70 = 7.571$. The other market information of this example comes from Table 2 and Table 5. The time to maturity is $T = 0.88493$. Dividend yields of two stocks are zeros.

That is, $q_A = q_B = 0.0\%$. Now we want to compute the value of exchange option $C_{B/A}\left(S_{B/A}^0, K_{B/A}\right)$ in equation (5) in Section 2. The unit of the call option value is the number of stock A.

$$
\begin{aligned}
&C_{B/A}(6.908, 7.571) \\
&\quad = (80.41)\left\{\max\left(\frac{745.23}{119.23} - 7.571, 0\right)(0.205) + \max\left(\frac{555.48}{119.23} - 7.571, 0\right)(0.011)\right. \\
&\qquad + \max\left(\frac{414.05}{119.23} - 7.571, 0\right)(0.021) + \max\left(\frac{745.23}{80.41} - 7.571, 0\right)(0.053) \\
&\qquad + \max\left(\frac{555.48}{80.41} - 7.571, 0\right)(0.464) + \max\left(\frac{414.05}{80.41} - 7.571, 0\right)(0.007) \\
&\qquad + \max\left(\frac{745.23}{54.23} - 7.571, 0\right)(0.0) + \max\left(\frac{555.48}{54.23} - 7.571, 0\right)(0.0) \\
&\qquad \left. + \max\left(\frac{414.05}{54.23} - 7.571, 0\right)(0.239)\right\} = 8.393 \qquad (19)
\end{aligned}
$$

Denominators in the equation, i.e. 119.23, 80.41, and 54.23, represent the price of stock A - See Figure 7 - and the numerators, i.e. 745.23, 555.48, and 414.05, the price of stock B - See Figure 8 - at each node after two time steps. Now apply Margrabe formula (6) to obtain the implied volatility of the barter trade (or exchange) terms B/A. The result is $\tilde{\sigma}_{B/A} = 0.1243$. From equation (8), the implied covariance is

$$
\widetilde{cov}(R_A, R_B) = \frac{\tilde{\sigma}_A^2 + \tilde{\sigma}_B^2 - \tilde{\sigma}_{B/A}^2}{2} = \frac{0.2961^2 + 0.2209^2 - 0.1243^2}{2} = 0.0605
$$

$$
\tilde{\rho}_{A,B} = \frac{\widetilde{cov}(R_A, R_B)}{\tilde{\sigma}_A \tilde{\sigma}_B} = \frac{0.0605}{0.2961 \times 0.2209} = 0.9252 \qquad \blacksquare
$$

More accurate examples are given in Section 5. There we consider longer time steps and consider more options for each stock. We also consider the coupling constraint $(18d)$ when we obtain the joint risk neutral probability by solving QP.

4. Application to Portfolio Selection

In this section we will apply the implied covariance obtained in the previous sections to the portfolio selection problem. However, we will be focused on a partial equilibrium analysis to learn more about what a change in the option premium does to the portfolio. To this end, first, we will first conduct a comparative static analysis at an initial equilibrium. For example, we will show the effect on the optimal weight of portfolio when the implied covariance changes. Next, we will show the effect on the efficient frontier of the portfolio when the option price changes.

Let's assume that the portfolio is constructed by n number of stocks and the return rates on each stock are given. New notations used in this section are as follows.

$\widetilde{\boldsymbol{R}} = \left(\tilde{R}_1, \cdots, \tilde{R}_n\right)^T$: the vector of the expected returns on stock $i, i = \{1, 2, \ldots, n\}$.

$\boldsymbol{\Omega}$: the variance/covariance matrix.

$\widetilde{\boldsymbol{\Omega}}$: the *implied* variance/covariance matrix.

$\boldsymbol{w} = (w_1, \cdots, w_n)^T$: the vector of portfolio weights of risky assets, where $\sum_{i=1}^{n} w_i = 1$.

$\boldsymbol{e}$: the $n \times 1$ unit column vector with the accounting identity, $\boldsymbol{w^T e} = \sum_{i=1}^{n} w_i$

At this time we assume that the implied expected returns ($\tilde{R}_i$'s) are given, but at the end of this section we will obtain them from the equilibrium market capitalization. We distinguish between $\boldsymbol{\Omega}$ and $\widetilde{\boldsymbol{\Omega}}$ for the purpose of comparative static analysis.

4.1 Comparative Static Analysis

Consider a mean-variance portfolio optimization problem (**POP**) by assuming a special class of the exponential utility function. Here γ is the coefficient of risk aversion.

$$Maximize\ \ U = \boldsymbol{w^T}\widetilde{\boldsymbol{R}} - \frac{\gamma}{2}\boldsymbol{w^T}\boldsymbol{\Omega}\boldsymbol{w} \tag{20a}$$

$$Subject\ to\quad \boldsymbol{w^T}\boldsymbol{e} = 1 \tag{20b}$$

$$\boldsymbol{\Omega} = \widetilde{\boldsymbol{\Omega}} \tag{20c}$$

Note that we added the constraint (20c) in order to analyze the effect on optimal portfolio weights when the implied covariance changes. The Lagrangian function corresponding to (20) is

$$\mathcal{L} = \boldsymbol{w^T}\widetilde{\boldsymbol{R}} - \frac{\gamma}{2}\boldsymbol{w^T}\boldsymbol{\Omega}\boldsymbol{w} + \lambda(1 - \boldsymbol{w^T}\boldsymbol{e}) + \sum_{i,j=1}^{n} v_{ij}\left(\widetilde{\Omega}_{ij} - \Omega_{ij}\right) \tag{21}$$

with $\lambda \in R,\ \boldsymbol{v} \in R^{n\times n}$. Here, λ and v_{ij}'s are Lagrange multipliers. Then the first order optimality conditions in POP are:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = \widetilde{\boldsymbol{R}} - \gamma\boldsymbol{\Omega}\boldsymbol{w} - \lambda\boldsymbol{e} = 0 \tag{22a}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} = -\frac{\gamma}{2}\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}} - \boldsymbol{v} = 0 \qquad (22b)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{e} = 0 \qquad (22c)$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{v}} = \widetilde{\boldsymbol{\Omega}} - \boldsymbol{\Omega} = 0 \qquad (22d)$$

The implied variance/covariance has its own *shadow* price, which can be derived from the result of equilibrium. That is, if the implied variance-covariance matrix $\widetilde{\boldsymbol{\Omega}}$ is invertible, the optimal portfolio weight is given by

$$\boldsymbol{w}^{*} = \gamma^{-1}\widetilde{\boldsymbol{\Omega}}^{-1}\left(\widetilde{\boldsymbol{R}} - \lambda^{*}\boldsymbol{e}\right) \qquad (23)$$

where $\lambda^{*} = \left(\sum_{i,j=1}^{n} \widetilde{\Omega}_{ij}^{-1} \widetilde{R}_j - \gamma\right) / \left(\sum_{i,j=1}^{n} \widetilde{\Omega}_{ij}^{-1}\right)$. At the optimality, the shadow price of the implied covariance $\widetilde{\Omega}_{ij}$ is $v_{ij}^{*} = -(\gamma/2)w_i^{*}w_j^{*}$.

It should also be noted that any variance-covariance matrix of population is positive semi-definite. Since, for any real vector $\boldsymbol{w} \in R^n$, the variance of any random variance is nonnegative, it follows that

$$\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Omega}\mathbf{w} = \sum_{i,j=1}^{n} w_i w_j \Omega_{ij} = \sum_{i,j=1}^{n} w_i w_j cov(R_i, R_j) = var\left(\sum_{i=1}^{n} w_i R_i\right) = var(\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{R}) \geq 0$$

Assuming that the second order optimality conditions are satisfied, See Appendix, we introduce the comparative static analysis at the point of the initial equilibrium. The effect of portfolio weight w_k^* when the implied covariance $\widetilde{\Omega}_{ij}$ changes is given by

$$\frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ij}} = \begin{cases} -\widetilde{\Omega}_{ij}^{-1} w_j^* (1 - w_j^*), & j = k \\ 0 & j \neq k \end{cases}, k = 1, \cdots, n;\ i, j = 1, \cdots, n. \tag{24}$$

See the Proof to this comparative statics in Appendix. It states that the higher the covariance with other stock, the lower the portfolio weight. Furthermore, as we can see below, the higher its own volatility, the lower the portfolio weight. If $\widetilde{\mathbf{\Omega}}$ is positive definite, then for any i, $\widetilde{\Omega}_{ii}^{-1} > 0$.[10] Now assume $0 < w_k^* < 1$. Then we have the following relation.

$$\frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ii}} = \begin{cases} -\widetilde{\Omega}_{ii}^{-1} w_k^* (1 - w_k^*) < 0, & i = k \\ 0 & i \neq k \end{cases} \tag{25}$$

Now we have the following, of which the second equality comes from the equation (24).

$$\frac{\partial w_k^*}{\partial C_i^0} = \sum_{j=1}^{n} \frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ij}} \cdot \frac{\partial \widetilde{\Omega}_{ij}}{\partial C_i^0} = \frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ik}} \cdot \frac{\partial \widetilde{\Omega}_{ik}}{\partial C_i^0}, \quad k = 1, \cdots, n;\ i = 1, \cdots, n. \tag{26}$$

From the properties of option pricing, we know $\partial C_i^0 / \partial \tilde{\sigma}_i^2 > 0$. And note that $\tilde{\sigma}_i^2 = \widetilde{\Omega}_{ii}$. Here, C_i^0 is the price of the call. From equation (26), we now derive the effect of a change in

[10] The inverse of any positive definite matrix is also positive definite. Therefore the diagonal elements of the inverse of any symmetric positive definite matrix are all positive.

option price on the portfolio weight. Since $0 < w_i^* < 1$ for some i, and noting that

$$\frac{\partial \widetilde{\Omega}_{ii}}{\partial C_i^0} = \left(\frac{\partial C_i^0}{\partial \widetilde{\Omega}_{ii}}\right)^{-1} > 0,$$

it follows that

$$\frac{\partial w_i^*}{\partial C_i^0} = \frac{\partial w_i^*}{\partial \widetilde{\Omega}_{ii}} \cdot \frac{\partial \widetilde{\Omega}_{ii}}{\partial C_i^0} < 0, \quad k = 1, \cdots, n; \ i = 1, \cdots, n \tag{27}$$

It states that the higher call option price, the lower the portfolio weight. Intuitively we can see this: First, the higher call option price, the higher the implied volatility. Then the higher the implied volatility, the lower the portfolio weight.

4.2 Effect on Efficient Portfolio Frontier

Now we want to know the effect on the efficient frontier of the portfolio when a certain option price changes. To this end, we consider the portfolio return $R_p = \sum_{i=1}^{n} w_i^* \tilde{R}_i$ and the portfolio standard deviation $\sigma_p = \sqrt{\sum_{i,j=1}^{n} \widetilde{\Omega}_{ij} w_i^* w_j^*}$ at the optimal weight $\boldsymbol{w}^*$. Then the pair (σ_p, R_p) is located at the efficient frontier on the $\sigma - R$ plane.

Now assume $0 < w_i^* < 1$ and $\tilde{R}_i > 0$ for some stock i. Then, by the chain rule, we can compute $\partial R_p / \partial C_i^0$ as follows.

$$\frac{\partial R_p}{\partial C_i^0} = \sum_{k=1}^{n} \frac{\partial R_p}{\partial w_k^*} \cdot \frac{\partial w_k^*}{\partial C_i^0} = \sum_{k=1}^{n} \tilde{R}_k \cdot \frac{\partial w_k^*}{\partial C_i^0} = \tilde{R}_i \cdot \frac{\partial w_i^*}{\partial C_i^0} < 0 \tag{28}$$

due to the result in equation (27). It states that the higher the call option price, the lower the portfolio return. Since $\widetilde{cov}(R_k, R_p) = \widetilde{cov}\left(R_k, \sum_{j=1}^{n} w_j^* R_j\right) = \sum_{j=1}^{n} \widetilde{\Omega}_{kj} w_j^*$, it is well known that

$$\sigma_p = \frac{\sigma_p^2}{\sigma_p} = \frac{\sum_{k,j=1}^{n} \widetilde{\Omega}_{kj} w_k^* w_j^*}{\sigma_p} = \sum_{k=1}^{n} w_k^* \left(\frac{\sum_{j=1}^{n} \widetilde{\Omega}_{kj} w_j^*}{\sigma_p} \right) = \sum_{k=1}^{n} w_k^* \cdot Risk(k) \tag{29}$$

Here $Risk(k) = \left(\frac{\widetilde{cov}(R_k, R_p)}{\sigma_p} \right)$. Equation (29) states that LHS σ_p can be regarded as the total weighted risks of all individual security in the portfolio. By inspection, we attribute the $Risk(k)$ to those risks arising from the individual asset's own volatility but also from the asset's covariance risks with other assets. Now assuming that $0 < w_i^* < 1$ and $Risk(i) > 0$, together with the result from equations (27) and (29), we have

$$\frac{\partial \sigma_p}{\partial C_i^0} = \sum_{k=1}^{n} \frac{\partial \sigma_p}{\partial w_k^*} \cdot \frac{\partial w_k^*}{\partial C_i^0} = \sum_{k=1}^{n} Risk(k) \left(\frac{\partial w_k^*}{\partial C_i^0} \right) = Risk(i) \left(\frac{\partial w_i^*}{\partial C_i^0} \right) < 0. \tag{30}$$

In (30), the second equality is satisfied since $\frac{\partial \sigma_p}{\partial w_k^*} = \frac{1}{2} \sigma_p^{-1} 2\left(\sum_{j=1}^{n} \widetilde{\Omega}_{kj} w_j^*\right) = \frac{\sum_{j=1}^{n} \widetilde{\Omega}_{kj} w_j^*}{\sigma_p} = Risk(k)$. Note that $Risk(i) > 0$ implies $\widetilde{cov}(R_i, R_p) > 0$. Therefore, under the conditions of $0 < w_i^* < 1$ and $\widetilde{cov}(R_i, R_p) > 0$, we have $\frac{\partial \sigma_p}{\partial C_i^0} < 0$. It states that the higher the call option price, the lower the portfolio standard deviation. Now assume that the option price of stock i decreases by ΔC_i^0. Then from equation (28) R_p increases by $\Delta R_p^{'}$ and σ_p also increases by $\Delta\sigma_p^{'}$ from equation (30). According to the quantities of $\Delta R_p^{'}$ and $\Delta\sigma_p^{'}$, the efficient frontier may be changed. Figure 9 shows the effect on the efficient frontier when the option price C_i^0 changes. As shown in the figure, if $\frac{\Delta R_p^{'}}{\Delta\sigma_p^{'}} > \frac{\Delta R_p}{\Delta\sigma_p^{'}}$, the efficient frontier moves to upward. And the optimal portfolio is changed and the Sharpe ratio is increased. Here ΔR_p is the change of portfolio return along the old efficient frontier when σ_p increases by $\Delta\sigma_p^{'}$.

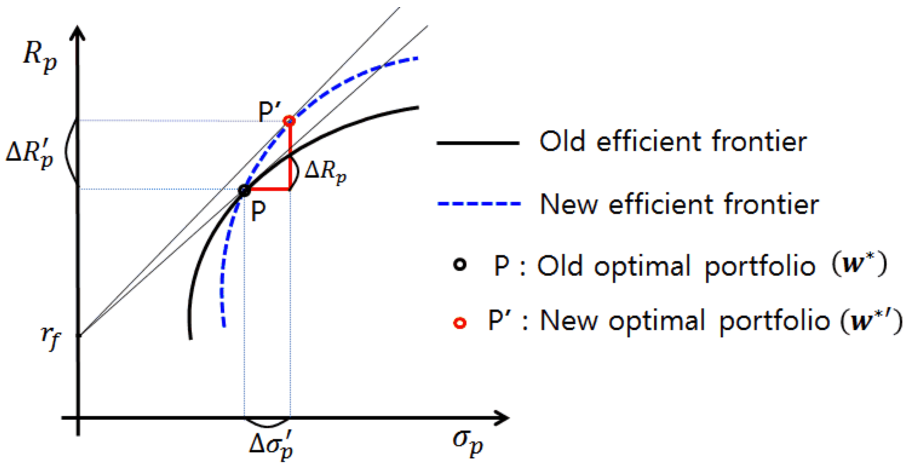


Figure 9. The effect on the efficient frontier when an option price changes

At this time we note that the implied expected return $\widetilde{\boldsymbol{R}}$ of our portfolio can be obtained from the market capitalization. Consider the following portfolio selection problem **(POP')** which includes the riskless asset. Denote $\boldsymbol{y}$ as the optimal portfolio weight in this case.

$$Maximize\ \ Z = \boldsymbol{y}^{\boldsymbol{T}}\widetilde{\boldsymbol{R}} + (1 - \boldsymbol{y}^{\boldsymbol{T}}\boldsymbol{e})r_f - \frac{\gamma}{2}\boldsymbol{y}^{\boldsymbol{T}}\widetilde{\boldsymbol{\Omega}}\boldsymbol{y} \tag{31}$$

Then we have the first order optimality condition as followers.

$$\frac{\partial Z}{\partial \boldsymbol{y}} = \mathbf{0} \;\Rightarrow\; \widetilde{\boldsymbol{R}} - r_f \boldsymbol{e} - \gamma\, \widetilde{\boldsymbol{\Omega}}\, \boldsymbol{y}^* = \mathbf{0} \tag{32}$$

By comparing $\boldsymbol{w}^*$ of the problem (POP) with the optimal market weights $\boldsymbol{y}^*$ of (POP'), we have the relationship between $\boldsymbol{w}^*$ and $\boldsymbol{y}^*$: $\boldsymbol{w}^* = \boldsymbol{y}^*/(\boldsymbol{y}^{*\boldsymbol{T}}\boldsymbol{e})$. That is, $\boldsymbol{w}^*$ is proportional to $\boldsymbol{y}^*$. Then we have $\widetilde{\boldsymbol{R}} - r_f \boldsymbol{e} = \bar{\gamma}\, \widetilde{\boldsymbol{\Omega}}\, \boldsymbol{w}^*$ with $\bar{\gamma}$ proportional to γ. Now in equilibrium, all investors should follow this optimal portfolio. That is, if the market is at the equilibrium, the optimal weight should be market weight: $\boldsymbol{w}^* = \boldsymbol{w}_{mkt}$. Inversely, if we know market weight $\boldsymbol{w}_{mkt}$, then we can obtain the implied expected return. So we have the following relationship for the implied expected return equilibrium.

$$\widetilde{\boldsymbol{R}} = \bar{\gamma}\, \widetilde{\boldsymbol{\Omega}}\, \boldsymbol{w}_{mkt} + r_f \boldsymbol{e} \tag{33}$$

The market weight $\boldsymbol{w}_{mkt}$ can be obtained from market capitalization. As we see the equation (33), $\bar{\gamma}$ is same for any stock. Thus $\bar{\gamma}$ is interpreted as the market price of risk.

5. Numerical Example

In this section, we only demonstrate how our results from the calculated implied variance-covariance matrix affect the portfolio numerically. Admittedly, conducting a fuller empirical study would be beyond the scope of this research.

We have considered four risky stocks, i.e. Amazon, Ebay, FaceBook, and Google; and a market index, e.g. S&P 500 index. We also have examined their respective European call-options. The yearly dividend rates of the stocks are all 0%. We used the end-of-day closing price for February 26, 2015. The maturity date of all call-options is January 15, 2016. We selected four exercise prices for each underlying asset, of which two are higher and two are lower than the current day stock price. To apply the binomial lattice, we have traced 10 time steps. Since the time to maturity is $T = 0.88493 = 323/365$, $\Delta t = T/10 = 0.08849$.

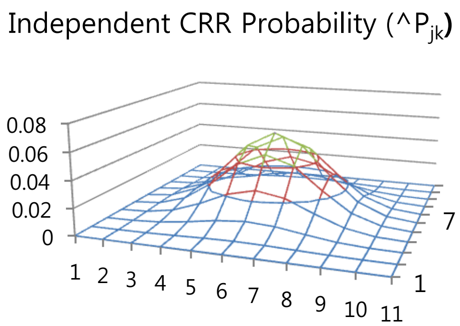
We solved QP, equations$(18a - i)$, for each pair of stocks and index. First, the pair of Amazon and Ebay, let P_{jk} be the implied joint probability obtained by solving the corresponding QP. We regard Amazon's stock price as the numeraire. Then we will compare CRR risk neutral joint probability $\hat{P}_{jk}$ when S_A^T and S_B^T are independent. It can be obtained by assuming that each stock price moves independently along a binomial lattice. To give a description of the binomial lattice for stock A, let p_A be the risk neutral probability of moving up to the next step, then the probability that S_A^T is at the j-th node after n steps (at maturity) is given by $\hat{P}_{j.} = \binom{n}{j} p_A^j (1 - p_A)^{n-j}$. [11] In the same manner for stock B, let p_B be the risk neutral probability of moving up to the next step, then the probability that S_B^T is at the k-th node after n steps (at maturity) is given by $\hat{P}_{.k} = \binom{n}{k} p_B^k (1 - p_B)^{n-k}$. Therefore the

[11] If the time length of one step is Δt and the volatility of stock A is σ_A, then $p_A = \frac{a_A - d_A}{u_A - d_A}$. Here, $a_A = e^{(r_f - q_A)\Delta t}$, $u_A = e^{\sigma_A \sqrt{\Delta t}}$ and $d_A = 1/u_A$. And $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.

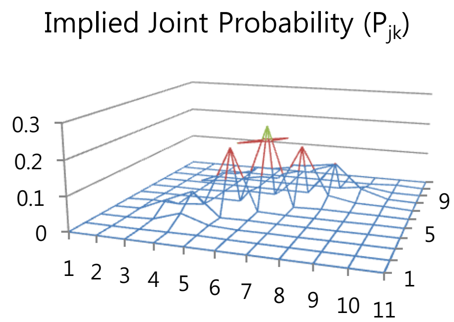
probability that (S_A^T, S_B^T) is at the joint node (j,k) at maturity is given by the following equation (34).

$$\hat{P}_{jk} = \hat{P}_{j.}\hat{P}_{.k} = \binom{n}{j} p_A^j (1-p_A)^{n-j} \binom{n}{k} p_B^k (1-p_B)^{n-k} \quad (34)$$

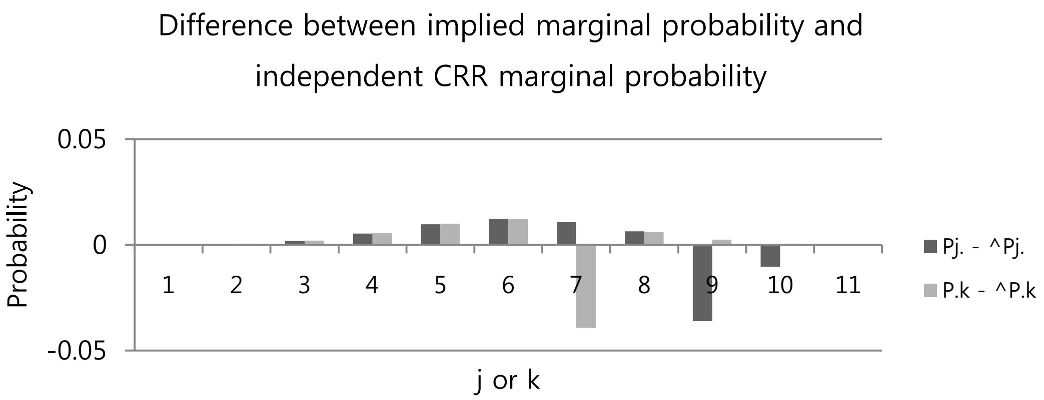
Figure 10 shows the results. The independent probability $\hat{P}_{jk}$ is computed by equation (34) and is plotted as Figure 10(a). The implied joint probability P_{jk} is obtained by solving QP (18a-j) and is plotted as Figure 10(b). The differences of two marginal probabilities are plotted as Figure 10(c).



(a)



(b)



(c)

Figure 10. (a) Independent probability ($\hat{P}_{jk}$) given by binomial lattice model. (b) Implied joint probability (P_{jk}) obtained by solving problem QP when Amazon and Ebay are considered. (c) Difference between the implied marginal probability and independent CRR marginal probability ($P_{j.} - \hat{P}_{j.}$) and ($P_{.k} - \hat{P}_{.k}$).

With the implied joint probability P_{jk}, we can compute the value of exchange call-option by applying the equation (5). Table 7 shows the results for each pair.

Table 7. The value of exchange European call-option

Value of B/A exchange European call option		B				
		Amazon	E-bay	Facebook	Google	S&P500
A	Amazon		4.653	6.314	20.752	63.390
	E-bay			5.140	1.461	25.240

Facebook				8.698	72.973
Google					67.608

If we apply the value of exchange call-option to the equation (6) of Theorem 2, then we can obtain the implied volatility of the exchange call-option. Table 8 shows the implied volatilities of exchange call-options.

Table 8. Implied volatilities of exchange European call-options

Implied volatility of B/A call-option		**B**				
		Amazon	E-bay	Facebook	Google	s&p500
A	Amazon		0.1160	0.1483	0.1526	0.1687
	E-bay			0.1933	0.0811	0.1560
	Facebook				0.1262	0.2215
	Google					0.1917

By applying the implied volatilities of two stocks to equations (8-9), we can compute the implied covariance and implied beta. Table 9 shows the implied covariance between each pair of stocks and the implied betas. The last column denotes the historical betas which can be obtained in Yahoo Finance.

Table 9. Implied covariance between each pair of stocks and implied betas

Implied covariance $\widetilde{cov}(R_A, R_B)$		B Amazon	E-bay	Facebook	Google	S&P500	Implied Beta β_A	Historical beta
A	Amazon	0.0927	0.0737	0.0792	0.0591	0.0414	2.220	1.27
	E-bay	0.0737	0.0681	0.0592	0.0552	0.0312	1.672	0.82
	Facebook	0.0792	0.0592	0.0877	0.0603	0.0286	1.534	0.85
	Google	0.0591	0.0552	0.0603	0.0488	0.0248	1.329	1.07

The market capitalization, the market portfolio weights, and the implied expected returns are given as Table 10. Here the implied expected return $\widetilde{\boldsymbol{R}}$ can be obtained by equation (33) when $\bar{\gamma} = 1.0$ and $r_f = 2.14\%$.

Table 10. Market portfolio weights and the implied expected excess return

	Amazon	E-bay	Facebook	Google
Market capitalization ($B)	173.79	71.89	220.93	378.09
Market portfolio weights $\boldsymbol{w}_{mkt}$	0.2057	0.0851	0.2616	0.4476
Implied expected return $\widetilde{\boldsymbol{R}}$	9.389%	8.253%	9.269%	7.585%

Now assume that the implied expected returns are those as Table 10. Then by the formula (23) or by the first order optimality conditions (22a-d) we have the optimal portfolio weights given as Table 11 when $\gamma = 0.75$ or $\gamma = 1.5$.

Table 11. Optimal portfolio weights

Risk aversion coefficient	Optimal portfolio weights $\boldsymbol{w}^*$				Sum	Optimal multiplier
	Amazon	E-bay	Facebook	Google		
$\gamma = 0.75$	0	0.14913	0	0.85087	1.0	0.12902
$\gamma = 1.5$	0	0.11243	0	0.88757	1.0	0.17135

The eigenvalues of the variance-covariance matrix of four stocks $\widetilde{\boldsymbol{\Omega}}$ in Table 9 are: 0.26354478, 0.02432448, 0.00552764, 0.00384648. Since all the eigenvalues are positive, $\widetilde{\boldsymbol{\Omega}}$ is positive definite. Therefore the second order sufficient condition is satisfied at the solution given by Table 11. Now we compute the effect of portfolio weight w_k^* when the implied covariance $\widetilde{\Omega}_{ij}$ changes. To do this, we compute $\widetilde{\boldsymbol{\Omega}}^{-1}$ as follows.

$$\widetilde{\boldsymbol{\Omega}}^{-1} = \begin{pmatrix} 65.748 & 35.464 & -69.706 & -25.998 \\ 35.464 & 89.4647 & -79.251 & -33.212 \\ -69.706 & -79.251 & 175.975 & -56.885 \\ -25.998 & -33.212 & -56.885 & 154.605 \end{pmatrix} \tag{37}$$

Then from equation (26) we have $\frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ij}}$ as follows. Since $\frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ij}} = 0$ for $j \neq k$, Table 12 shows $\frac{\partial w_j^*}{\partial \widetilde{\Omega}_{ij}}$. For example, if $\widetilde{\Omega}_{12}$ increases by Δ, then w_2^* will increase by -0.46998Δ and w_4^* will increase by 0.96503Δ. In this case, w_1^* and w_3^* will not change. When we read the diagonal elements of the Table 12, we find $\frac{\partial w_i^*}{\partial \widetilde{\Omega}_{ii}}$. That is, $\frac{\partial w_1^*}{\partial \widetilde{\Omega}_{11}} = 0$, $\frac{\partial w_2^*}{\partial \widetilde{\Omega}_{22}} = 0.77541, \frac{\partial w_3^*}{\partial \widetilde{\Omega}_{33}} = 0$, $\frac{\partial w_4^*}{\partial \widetilde{\Omega}_{44}} = 1.12440$.

Table 12. The effect on the optimal portfolio weight when covariance changes ($\gamma = 0.75$)

	$\frac{\partial w_j^*}{\partial \widetilde{\Omega}_{ij}}$	***j*** Amazon	E-bay	Facebook	Google
	Optimal weight w_j^*	0.0	0.14913	0.0	0.85087
i	Amazon	0.0	-0.46998	0.0	0.96503
	E-bay	0.0	0.77541	0.0	-1.27078
	Facebook	0.0	0.74455	0.0	-0.78033
	Google	0.0	-1.22734	0.0	1.12440

6. Summary and Conclusions

The value of derivatives depends on the value and the volatility of the underlying assets. Ever since the renowned Black-Scholes option pricing formula, many financial economists and practitioners have taken interests in discovering the nature of the probability distribution for the return on the underlying assets embedded in option prices. However, the implied volatility turned out not to be unique due to its own smiles, term structures, and surfaces, etc. As a result, people questioned about the validity of assuming the log-normality for the underlying assets pointing out that stock returns are subject to frequent jumps and discontinuities.[12] This paper

[12] However, what if the returns for the underlying stocks are *ex ante* normal and all that matters is the *ex ante* returns? In addition, it is widely known that stocks with fractal dimension exceeding 1.5 tend to be anti-persistent while returns on stocks with fractal dimension less than

has suggested a way to find a unique value of the implied variances and covariances implicit in option prices, which has been smoothed.[13] We have taken a non-parametric programming approach to produce a smooth overall value of the probability parameters just as Prof. Rubinstein once outlined in his 1994 AFA presidential address.

Obviously, in equilibrium, the expected return on component securities in a portfolio is a function of implied variance and covariances of individual securities. However, these variance and covariances are endogenously determined, which can be imputed from the regular and/or exchange (or pseudo exchange) options premium; and so must be true with the implied expected return on stocks to the extent that the implied volatility of the market portfolio enters into the equilibrium return equation for individual assets. The end result is that it must impact the portfolio frontier in the usual portfolio selection problems.

A new numerical procedure presented in this paper to compute the implied return covariances has assumed no arbitrage in an efficient capital market, where we inferred the value of all possible barter trades. We have shown that it is possible to find risk neutral bivariate

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1.5 are persistent, which is generally known as *momentum*. Thus, in some cases, a sudden jump in prices will return back to normal and some stocks that have just jumped will continue on its own momentum. However, random walk is often characterized in terms of 1.5 fractal dimension. Thus, all jump phenomenon may be seen temporary so that all jumps can be conveniently ignored.

[13] See Footnote 10 for a possible justification.

probabilities through a programming technique from various stock and/or stock exchange options with differing strike prices. We have then shown that one can further compute the implied covariances between any two assets in a barter exchange trade from the value of any pseudo exchange options, even when exchange options do not exist in the marketplace. We have presented some differences and the robustness between our model and all other previous models introduced by Cox, Ross and Rubinstein (CRR), *op cit*, and Rubinstein, *op cit*. Most importantly, our procedure is not only applicable to the case of a single underlying asset but also to any other options contract involving more than one asset.

In closing, we are compelled to state that our discussions in Section 4 are only a *partial* equilibrium analysis nonetheless, where many things are held constant, when in reality, the implied volatility and covariances as well as the expected return on securities; and furthermore, the market portfolio weights must be simultaneously determined. This warrants further general equilibrium analysis. However, what is encouraging at the moment is that we may have found ways to compute forward looking covariances as opposed to historical covariances, which requires the use of historical sample data. It is hoped that our methods can be utilized in portfolio studies in the future.

APPENDICES

A. Second Order Optimality Conditions:

Theorem 5. We now argue that if the implied variance-covariance matrix $\widetilde{\boldsymbol{\Omega}}$ is positive definite, the second order sufficient optimality condition is satisfied at the optimal solution vector $(\boldsymbol{w}, \boldsymbol{\Omega}, \lambda, \boldsymbol{v}) = (\boldsymbol{w}^*,\ \widetilde{\boldsymbol{\Omega}}, \lambda^*, \boldsymbol{v}^*)$.

Proof. To prove, we need the following two preliminary results.

Proposition 1 The quadratic form of the matrix which consists of the second partial derivatives of $\mathcal{L}$ with respect to $\mathbf{w}$ and $\boldsymbol{\Omega}$ has same value regardless of order of w and Ω. Technically,

$$\boldsymbol{w}^T \frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{\Omega} \partial \boldsymbol{w}} \boldsymbol{\Omega} = \boldsymbol{\Omega} \frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{w} \partial \boldsymbol{\Omega}} \boldsymbol{w} = -\gamma \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\Omega} \boldsymbol{w}.$$ [14]

Proof. To compute $\boldsymbol{w}^T \frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{\Omega} \partial \boldsymbol{w}} \boldsymbol{\Omega}$, we need to compute $\frac{\partial}{\partial \Omega}(\boldsymbol{\Omega w})$ in advance.

Since $\frac{\partial}{\partial \Omega_{ij}}(\boldsymbol{\Omega w}) = \frac{\partial}{\partial \Omega_{ij}} \begin{pmatrix} \Omega_{11} w_1 + \Omega_{12} w_2 + \cdots + \Omega_{1n} w_n \\ \Omega_{21} w_1 + \Omega_{22} w_2 + \cdots + \Omega_{2n} w_n \\ \vdots \\ \Omega_{n1} w_1 + \Omega_{n2} w_2 + \cdots + \Omega_{nn} w_n \end{pmatrix}^T = e_i^T w_j$, where $\boldsymbol{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \cdot \\ \cdot \\ i^{th}, \\ \cdot \\ \cdot \end{matrix}$

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[14] $\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{z}}$ where $\boldsymbol{z}^{\boldsymbol{T}} = (\boldsymbol{\Omega}_{\boldsymbol{1}.}, \cdots, \boldsymbol{\Omega}_{\boldsymbol{n}.})$ and $\boldsymbol{\Omega}_{\boldsymbol{i}.}$ is the i-th row of $\boldsymbol{\Omega}$.

$$\frac{\partial(\boldsymbol{\Omega w})}{\partial \boldsymbol{\Omega}} = \begin{pmatrix} \boldsymbol{e_1^T} w_1 & \cdots & \boldsymbol{e_1^T} w_n \\ \vdots & \ddots & \vdots \\ \boldsymbol{e_n^T} w_1 & \cdots & \boldsymbol{e_n^T} w_n \end{pmatrix} \equiv \boldsymbol{A} \quad \text{where } \mathbf{A} \text{ is an } n \times n^2 \text{ matrix.}$$

This means that

$$\boldsymbol{w^T A \Omega} = \sum_{i,j=1}^{n} w_i\, \boldsymbol{e_i^T} w_j \boldsymbol{\Omega_{.j}} = \sum_{i,j=1}^{n} w_i w_j \Omega_{ij} = \boldsymbol{w^T \Omega w} \qquad \text{(A1)}$$

where $\boldsymbol{\Omega_{.j}}$ is the j^{th}column of $\boldsymbol{\Omega}$. From equation (22a) we have

$$\boldsymbol{w}^T \frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{\Omega} \partial \boldsymbol{w}} \boldsymbol{\Omega} = -\gamma \boldsymbol{w^T \Omega w} \qquad \text{(A2)}$$

To compute $\boldsymbol{\Omega} \frac{\partial^2 \mathcal{L}}{\partial w \partial \boldsymbol{\Omega}} \boldsymbol{w}$, we need to compute $\frac{\partial}{\partial \boldsymbol{w}}(\boldsymbol{w^T w})$ in advance.

$$\frac{\partial}{\partial \mathrm{w}_i}(\boldsymbol{w^T w}) = \begin{pmatrix} 0 & \dots & w_1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ w_1 & \cdots & 2w_i & \cdots & w_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & w_n & \dots & 0 \end{pmatrix} \begin{matrix} . \\ . \\ i^{th} \\ . \\ . \end{matrix} \equiv \boldsymbol{B_i} \quad \text{where } \boldsymbol{B_i} \text{ is the } n \times n \text{ matrix.}$$

By letting $\mathbf{B} = \begin{pmatrix} \boldsymbol{B_1} \\ \vdots \\ \boldsymbol{B_n} \end{pmatrix}$, we have

$$\boldsymbol{\Omega^T B w} = \sum_{i,j=1}^{n} \boldsymbol{\Omega_{i.}} (\mathbf{B}_i)_{.j} w_j = 2 \sum_{i,j=1}^{n} w_i w_j \Omega_{ij} = 2\boldsymbol{w^T \Omega w} \qquad \text{(A3)}$$

where $(\mathrm{B}_i)_{.j}$ is the j^{th}column of the matrix B_i. Therefore from the equation (22b) we have

$$\boldsymbol{\Omega}^T \frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{\Omega} \partial \boldsymbol{w}} \boldsymbol{w} = -\gamma \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\Omega} \boldsymbol{w} \tag{A4}$$

■

Proposition 2 The quadratic form of the matrix which consists of the second partial derivatives of $\mathcal{L}$ with respect to $\boldsymbol{v}$ and $\boldsymbol{\Omega}$ has same value regardless of order of $\boldsymbol{v}$ and $\boldsymbol{\Omega}$. Technically, $\boldsymbol{\Omega}^{\boldsymbol{T}} \frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{v} \partial \boldsymbol{\Omega}} \boldsymbol{v} = \boldsymbol{v}^T \frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{\Omega} \partial \boldsymbol{v}} \boldsymbol{\Omega} = -\sum_{i,j=1}^{n} \Omega_{ij}\, v_{ij}$.

Proof. From equation (22b) and (22d), we have

$$\frac{\partial}{\partial \boldsymbol{v}}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}}\right) = -\frac{\partial}{\partial \boldsymbol{v}}(\boldsymbol{v}) \text{ and } \frac{\partial}{\partial \Omega}\left(\frac{\partial}{\partial \boldsymbol{v}}\right) = -\frac{\partial}{\partial \Omega}(\boldsymbol{\Omega}).$$

Noting that $\frac{\partial}{\partial \boldsymbol{v}}(\boldsymbol{v}) = \frac{\partial}{\partial \Omega}(\boldsymbol{\Omega})$, we now have

$$\frac{\partial \Omega}{\partial \Omega_{ij}} = \frac{\partial v}{\partial v_{ij}} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \begin{matrix} \cdot \\ \cdot \\ i^{th} \\ \cdot \\ \cdot \end{matrix} \equiv \boldsymbol{I}_{ij}$$

where $\boldsymbol{I}_{ij}$ is the $n \times n$ matrix with one at the (i,j)-position and zero otherwise.

By letting $\mathbf{M} = \begin{pmatrix} \boldsymbol{I}_{11} & \cdots & \boldsymbol{I}_{1n} \\ \vdots & \ddots & \vdots \\ \boldsymbol{I}_{n1} & \cdots & \boldsymbol{I}_{nn} \end{pmatrix}$, $\mathbf{M}$ is an $n^2 \times n^2$ matrix, and we finally have

$$\boldsymbol{\Omega}^{\boldsymbol{T}}\mathbf{M}\mathbf{v} = \sum\nolimits_{i,j=1}^{n} \boldsymbol{\Omega}_{\boldsymbol{i}.}\boldsymbol{I}_{\boldsymbol{ij}}\boldsymbol{v}_{\boldsymbol{j}.} = \sum\nolimits_{i,j=1}^{n} \Omega_{ij}v_{ij} \tag{A5}$$

Similarly, we have $\boldsymbol{v}^{\boldsymbol{T}}\mathbf{M}\boldsymbol{\Omega} = \sum_{i,j=1}^{n} \Omega_{ij}v_{ij}$. ■

Now we present the following Hessian matrix to prove Theorem 5. The matrix is,

$$\begin{array}{cccc} \mathbf{w} & \boldsymbol{\Omega} & \lambda & \boldsymbol{v} \end{array}$$

$$\mathbf{H} = \begin{pmatrix} -\gamma\boldsymbol{\Omega} & -\gamma\boldsymbol{A} & -\boldsymbol{e} & \mathbf{0} \\ -(\gamma/2)\boldsymbol{B} & \mathbf{0} & \mathbf{0} & -\boldsymbol{M} \\ -\boldsymbol{e}^{T} & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{M} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{A6}$$

Here, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{M}$ are defined in the proofs of the Propositions 1-2. From Proposition 1, we have

$$-\gamma\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{\Omega} - (\gamma/2)\boldsymbol{\Omega}\boldsymbol{B}\boldsymbol{w} = -2\gamma\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Omega}\mathbf{w} \tag{$A7$}$$

From the equation (22b), we have $\boldsymbol{v} = -\frac{\gamma}{2}\boldsymbol{w}\boldsymbol{w}^{\boldsymbol{T}}$, that is $v_{ij} = w_i w_j$. Therefore we have

$$\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{M\Omega} = \sum_{i,j=1}^{n} \Omega_{ij} v_{ij} \text{ (from Proposition 2)}$$

$$= -(\gamma/2)\sum_{i,j=1}^{n} \Omega_{ij} w_i w_j$$

$$= -(\gamma/2)\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Omega w} \qquad (A8)$$

For any $\mathbf{z} = \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\Omega} \\ \lambda \\ \boldsymbol{v} \end{pmatrix} \in R^n \times R^{n\times n} \times R \times R^{n\times n}$, compute $\boldsymbol{z}^{\boldsymbol{T}}\boldsymbol{Hz}$. Here, $\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{w} = \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{e} = 1$ from equation (22d). Then we have the following by (A7) and (A8).

$$\boldsymbol{z}^{\boldsymbol{T}}\boldsymbol{Hz} = -\gamma \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Omega w} + \{-\gamma \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{A\Omega} - (\gamma/2)\boldsymbol{\Omega B w}\} + \{-\lambda e^T w - \lambda w^T e\} - 2\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{M\Omega}$$

$$= -\gamma \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Omega w} - 2\gamma \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Omega w} - 2\lambda + \gamma \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\Omega w}$$

Therefore at the optimal solution $\boldsymbol{z}^* = \left(\boldsymbol{w}^*, \widetilde{\boldsymbol{\Omega}}, \lambda^*, \boldsymbol{v}^*\right)$, we have the following second order sufficiency.

$$\boldsymbol{z}^{*\boldsymbol{T}}\boldsymbol{Hz}^* = -2\gamma \boldsymbol{w}^{*\boldsymbol{T}}\widetilde{\boldsymbol{\Omega}}\boldsymbol{w}^* < 0 \qquad (A9)$$

for the fact that $\widetilde{\boldsymbol{\Omega}}$ is positive definite by the hypothesis of the theorem. ■

B. Comparative Statics: Effects of Implied Covariance Risks on Portfolio Selection

We stated earlier that if the implied variance-covariance matrix $\widetilde{\boldsymbol{\Omega}}$ is invertible, the effect of portfolio weight w_k^* when the implied covariance $\widetilde{\Omega}_{ij}$ changes is given by

$$\frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ij}} = \begin{cases} -\widetilde{\Omega}_{ij}^{-1} w_j^* (1 - w_j^*), & j = k \\ 0 & j \neq k \end{cases}, k = 1, \cdots, n;\ i, j = 1, \cdots, n. \qquad (A10)$$

See equation (24).

Proof. To prove that equation (A10) is true, compare $\boldsymbol{w}^*$ of the problem (**POP**) with the optimal weights $\boldsymbol{y}^*$ of the following portfolio selection problem (**POP'**), which includes the riskless asset.

$$Maximize\ \ Z = \boldsymbol{y}^T \widetilde{\boldsymbol{R}} + (1 - \boldsymbol{y}^T \boldsymbol{e}) r_f - \frac{\gamma}{2} \boldsymbol{y}^T \widetilde{\boldsymbol{\Omega}} \boldsymbol{y}$$

The first order optimality condition is

$$\frac{\partial Z}{\partial \boldsymbol{y}} = 0 \ \Rightarrow\ \widetilde{\boldsymbol{\Omega}} \boldsymbol{y}^* = \gamma^{-1} (\widetilde{\boldsymbol{R}} - r_f \boldsymbol{e}) \qquad (A11)$$

By the portfolio separation theorem, we have the following relationship between $\boldsymbol{w}^*$ and $\boldsymbol{y}^*$.

$$\boldsymbol{w}^* = \boldsymbol{y}^* / ({\boldsymbol{y}^*}^T \boldsymbol{e});\quad \text{or}\ \ w_k^* \left(\sum_{l=1}^{n} y_l^* \right) = y_k^*, \qquad k = 1, \cdots, n \qquad (A12)$$

Partially differentiating $\partial \widetilde{\Omega}_{ij}$ on both sides of the equation (A11), we have

$$\frac{\partial y_k^*}{\partial \widetilde{\Omega}_{ij}} = \begin{cases} -\widetilde{\Omega}_{ij}^{-1} y_j^*, & j = k \\ 0 & j \neq k \end{cases}, k = 1, \cdots, n; \ i, j = 1, \cdots, n \qquad (A13)$$

By partially differentiation of $\partial\widetilde{\Omega}_{ij}$ on both sides of the equation (A12), when $j = k$, we also have

$$\frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ij}} \left(\sum\nolimits_{l=1}^{n} y_l^*\right) - w_k^* \left(-\widetilde{\Omega}_{ij}^{-1} y_j^*\right) = -\widetilde{\Omega}_{ij}^{-1} y_k^* \,. \qquad (A14)$$

Therefore we have the following equation.

$$\frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ij}} = \frac{-\widetilde{\Omega}_{ij}^{-1} y_k^* (1 - w_k^*)}{\sum_{l=1}^{n} x_l^*} \qquad (A15)$$

Since $\frac{y_k^*}{\sum_{l=1}^{n} y_l^*} = w_k^*$ from equation (A12), we have the equation (A10). Clearly, $\frac{\partial w_k^*}{\partial \widetilde{\Omega}_{ij}} = 0$ for $j \neq k$. ■

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