How Annuity Demand is Affected by Insurer Default Risk

Bong-Gyu Jang*, Hyeng Keun Koo†, and Seyoung Park‡

*Department of Industrial and Management Engineering, POSTECH, Korea. Tel: +82-54-279-2372, Fax: +82-54-279-2870, E-mail: bonggyujang@postech.ac.kr

†Department of Business Administration, Ajou University, Korea. Tel: +82-31-219-2706, E-mail: koo_h@msn.com

‡Corresponding author: The Credit Finance Research Institute, The Credit Finance Association of Korea, and Department of Industrial and Management Engineering, POSTECH, Korea. Tel: +82-2-2011-0677, Fax: +82-2-2011-0701(2), E-mail: seyoungpark86@gmail.com

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Abstract

This paper considers a model of optimal consumption and portfolio choice of the retired person who has partially annuitized her wealth and faces default risk of her annuity provider. We develop a new method for solving the optimal consumption and portfolio choice problem in an incomplete financial market. We find that the effects of default risk on the retiree's optimal investment strategies and annuity demand depend crucially on the level of default risk, risk aversion, wealth, and investment opportunity. Higher default risk may lead retired people to increase their stock holdings. Lower default risk can substantially increase the demand for annuities. The annuity demand increases as risk aversion increases if default risk is small, whereas the annuity demand decreases as risk aversion increases if default risk is large.

Keywords: default risk, annuity demand, retiree, optimal consumption, optimal portfolio

1 Introduction

There has been an increasing concern about the soundness of corporate and public pension systems in major developed countries. Corporations as well as federal, state, municipal and other local governments are expected to face difficulties in providing for pensions due to their increasing funding problems. The 2008 global crisis had a significant negative impact on the financial positions of the pension sector. According to the pension fund return statistics in selected OECD countries (Figure 1), most countries experienced negative returns on their public pension funds in economic downturns. The fiscal pressure on the public pension system also increased due to demographic trends; life expectancy increased and fertility rate decreased. The support ratio, the ratio of workers to pensioners, has declined in most of the developed world.[1] Without further legislation or change in benefits, the fiscal deficits are expected to increase continuously. A major cut in pension benefits ahead took place in the U.S. on July 18, 2013, the city of Detroit, Michigan filed for Chapter 9 bankruptcy and consequently, proposed pension benefit cuts. A deal to cut monthly pension benefits for thousands of retirees by 4.5% was unanimously endorsed by a Detroit pension board on April 16, 2014.[2]

In this paper we study the effect of an unexpected, exogenous and permanent reduction in future annuity payments on annuity demand of a retired individual. The reduction is likely to happen in public plans due to the political risk of benefit cuts. With regard to the private annuity products sold by insurance companies, the risk of default of an insurance provider is negligible; in the U.S. and other jurisdictions, there exists guarantee funds which protect annuitants against default of annuity providers.[3] Annuities are useful hedging tools

[1] In the U.S. the number of workers paying into the pension program was 5.1 per retiree in 1960. However, in 2007 the number decreased to 3.3. This is expected to decrease to 2.1 by 2035. For the details, see the 2010 annual report of the board of trustees of the federal old-age and survivors insurance and federal disability insurance trust funds.

[2] There are other examples of bankruptcy in the U.S.: Jefferson County, Alabama filed for bankruptcy on November 9, 2011, and Stockton, California filed for Chapter 9 bankruptcy on June 28, 2012.

[3] In the U.S. National Organization of Life and Health Guaranty Associations (NOLHGA) offers guaranteed coverage for life annuities when an insurance company defaults. While the maximum coverage limits vary from state to state, most states offer $100,000 coverage in withdrawal and cash values for annuities. Actually, in case the claim of a policyholder exceeds the guaranty association coverage, the recovery amount of her excess claim depends highly on the liquidation ratio for the insolvency. We provided the details of how the state guaranty associations actually work in Appendix.

against longevity risk, which is any potential risk related to the increasing an individual's life expectancy. Thus, even a slight risk of a permanent decrease in future benefits is troublesome for a retiree who plans on securing financial resources in preparation for a long life.

Two extant works recently studied similar issues related to the effects of potential benefit cuts on the behavior of retirees. Lopes and Michaelides (2007) studied a model with defaultable annuities and concluded that significantly low annuity take-up cannot be explained by the possibility of rare events such as the default of the annuity provider. They asserted that although it is necessary to assume high risk aversion for a rare event to change behavior of a retiree, high risk aversion makes annuities more attractive as insurance against mortality risk. As a result, a rare event is unlikely to provide an explanation for the low annuity demand. Babbel and Merrill (2007) also made a contribution to this field of study in asserting that the participation in the annuity market may be low when retirees are exposed to default risk of annuity providers. Our end state is to provide a reconciliation of the seemingly contradictory conclusions reached by the two research teams. We show that the retiree's annuity demand increases as her risk aversion increases if default risk is small, which is consistent with Lopes and Michaelides' finding. We also show that the demand decreases with risk aversion if default risk is large, which is consistent with Babbel and Merrill's result.

We utilize two concepts to measure the effects of default risk of the annuity provider on the retiree's annuity demand: *implicit value of life annuity* and *certainty equivalent wealth gain (CEWG)*. The implicit value of life annuity is the marginal rate of substitution between a retiree's annuity holdings and financial wealth, so it is the retiree's subjective marginal value of her annuity holdings, i.e., her reservation price of an annuity when she is offered to purchase or sell a small amount of the annuity. It is also a proxy for the retiree's annuity demand; a higher value than the market price implies that the retiree is willing to buy it, but a lower value than the market price implies the opposite, she would be willing to sell it if she were given an opportunity to do so. We define the CEWG as the largest wealth that the retiree is willing to give up in exchange for making the annuity default-free. Thus, the CEWG serves as compensation for the retiree in return for bearing default risk of life annuity.

The main contribution of this paper is to show that the effects of default risk on the retiree's optimal investment strategies and annuity demand depend crucially on the level of default risk, risk aversion, wealth, and investment opportunity. We establish the following facts by using numerical examples:

- Higher default risk may lead retired people to increase their stock holdings.
- Lower default risk can substantially increase the demand for annuities.
- The annuity demand increases as risk aversion increases if default risk is small, whereas the annuity demand decreases as risk aversion increases if default risk is large.

1.1 Literature Review and Outline

It is well-known fact that, despite the theoretical appeal of annuities,[4] the actual annuity market is very thin. For instance, Inkmann *et al.* (2010) note that fewer than 6% of households participate in the annuity market. This is the so-called *annuity puzzle.* There has been numerous attempts to resolve the apparent contradiction between theoretical appeal of annuities and a thin annuity market. Mitchell *et al.* (1999) and Finkelstein and Poterba (2002, 2004) emphasize that the market price of annuities is higher than their actuarially-fair price due to the adverse selection of the annuity buyers. Friedman and Warshawsky (1990), Brown (2001), and Johnson *et al.* (2004) propose that the strong bequest motive of annuity buyers can generate low annuity demand. Sinclair and Smetters (2004) show that it is suboptimal for an individual who faces health shocks to annuitize all her wealth. Social security benefits and defined benefit pension plans are substitutes for annuitization (Dushi and Webb, 2004). Brown (2007) concludes that the low take-up rate of annuities cannot be resolved by any one factor, and a combination of adverse selection, inflation, and fees is necessary to explain it. Chalmers and Reuter (2012) demonstrate that the demand for life annuities has nothing to do with the level of life annuity prices and depends mainly on changes in individual characteristics and measures of investor sentiment. Brown (2009) and Benartzi *et al.* (2011) contend that behavioral factors, e.g., financial illiteracy, and behavioral biases can be reasons for under-annuitization.

In the literature regarding consumption and portfolio selection, Merton (1969, 1971) study the optimal consumption and portfolio choice problem without longevity risk. Richard (1975)

[4]In the early stage of research concerning private annuity markets, Yaari (1965) demonstrates that it is optimal for individuals without bequest motive to annuitize all of their wealth under an uncertain lifetime. Following Yaari's (1965) work, Davidoff *et al.* (2005) find much weaker sufficient conditions for the full annuitization than those imposed by Yaari (1965).

considers the problem with longevity risk and concluded that the full annuitization is optimal. In addition, there is an extensive literature which employs the portfolio-based model and investigates individuals' optimal investment behavior in the presence of an annuity market. In particular, a few researchers investigate consumption and portfolio selection in an incomplete financial market where income risk or mortality/longevity risk cannot be diversified away. For instance, Cairns *et al.* (2006) derive optimal asset allocation strategies in the presence of non-hedgeable salary risk, and Milevsky and Young (2007) investigate a retiree's optimal consumption, investment and annuitization policies when she faces a random time of death. Bayraktar *et al.* (2009) introduce a robust theory of individual's mortality risk valuation in an incomplete market. None of the existing literature, however, has treated the consumption and portfolio selection problem of a retiree in the presence of a non-diversifiable default risk of annuity provider.

We consider a model of optimal consumption and portfolio choice of the retired person who has partially annuitized her wealth and faces default risk of her annuity provider. More specifically, the retiree has already purchased an annuity product from the annuity provider, but her wealth is not yet fully annuitized. An annuitization is an irreversible transaction due to adverse selection, and in most cases the annuitization must take place near retirement. Milevsky and Young (2007) investigate the optimal annutization under such institutional all-or-nothing arrangements. In this paper, the all-or-nothing annuitization is not investigated. Instead, *given an already established annuity position* we explore how future annuity demand is affected by an unexpected, exogenous and permanent reduction in future annuity payments.[5] The annuity demand is aptly captured by *the implicit value of life annuity*. Because we model the default risk of insurer as a non-diversifiable and externally-driven random shock, the financial market consisting of the securities and insurance markets is essentially *incomplete*. We develop a new method for solving the optimal consumption and portfolio choice problem in such an incomplete financial market.[6]

[5] An interesting extension of this paper is to consider an open market structure in which a retiree is allowed to buy annuities in lump sums or continuously. Milevsky and Young (2007) consider the case where individuals can annuitize any fraction of their wealth at any time.

[6] Bensoussan *et al.* (2013) develop a new convex-duality approach for solving an optimal retirement problem (or an optimal stopping problem) for an individual in the presence of exogenously-driven unemployment risk. Our method shares the same fundamental idea with theirs. The default event of the annuity provider in our paper is assumed to have an exponential distribution, while the existing models dealing with income risk

Our paper is organized as follows. In Section 2 we describe a financial market in the presence of forced default events of the annuity provider and explain our problem. Section 3 provides analytical results concerning the optimal policies of the retiree, and Section 4 shows the implications of our model through numerical results. More specifically, we analyze the effects of the default risk on the retiree's optimal consumption and risky investment strategies, and the annuity demand. We check the robustness of our model in Section 5 and conclude in Section 6.

2 The Model

2.1 Financial Market

We consider the optimal consumption and portfolio selection problem of an economic agent who has retired from work (a retiree). The retiree's objective is to maximize the following lifetime utility:

$$U = E\Big[\int_0^{\tau_M} e^{-\beta t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt\Big], \tag{1}$$

where E is the expectation taken at time 0, c_t is per period consumption, τ_M is the time of the retiree's death, $\beta > 0$ is the retiree's subjective discount rate, and $\gamma > 0$ is her coefficient of absolute risk aversion. In (1) we assume that the retiree has constant absolute risk aversion (CARA). The assumption allows us to derive optimal strategies in closed form.[7] We assume that the time of the retiree's death, τ_M, is distributed according to the exponential distribution with intensity ν. In reality, the mortality rate is increasing over time, however, the constant mortality rate assumption is made for parsimony of the model. In this paper, we further assume that the retiree has no bequest motive. We know that the presence of the bequest motive reduces the motive to buy annuities (Friedman and Warshawsky 1990,

usually employ a Gaussian process, e.g., an arithmetic Brownian motion. For instance, Bayraktar *et al.* (2009) develop a martingale method for solving the portfolio choice problem with income risk, which is modeled as a Brownina motion. We technically contribute to the literature by developing a dynamic programming method to solve the incomplete market problem when the default event is distributed according to an exponential distribution.

[7]In our problem we assume the exponential utility function. A consumption and portfolio optimization problem in an incomplete market with exponential utility often admits a closed-form solution (Svensson and Werner, 1993). The method we propose, however, is applicable not only for the case where the utility function is exponential but for more general functions (Bensoussan *et al.*, 2013).

Brown 2001, and Johnson *et al.* 2004), so addition of the bequest motive will only reinforce our conclusion. In fact, the relaxation of the assumption does not change our main result.[8] By the assumption of the exponential distribution for τ_M, the retiree's utility function can be restated as

$$U = E\Big[\int_0^{\tau_M} e^{-\beta t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt\Big] = E\Big[\int_0^{\infty} e^{-(\beta+\nu)t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt\Big] \tag{2}$$

Thus, the utility function is equal to that of an infinitely-lived person with her subjective discount rate being increased by the mortality rate ν.

The retiree can trade securities. The securities market consists of two assets: a bond (or a risk-free asset) and a stock (or a risky asset). The bond price B_t follows

$$dB_t = rB_t dt,$$

where $r > 0$ is the risk-free interest rate, and the stock price S_t evolves according to the following geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu > r$ is the expected rate of the stock return, $\sigma > 0$ is the volatility of the return on the stock, and W_t is a standard Brownian motion defined on an appropriate probability space. μ and σ are equal to the mean and the standard deviation of the return on the stock, respectively, and summarizes the investment opportunity provided by the stock, i.e., they represent the expected return and risk in the market. We assume that r, μ, σ are constant. The assumption of a geometric Brownian motion for the stock price is standard in the literature on investment (e.g., Merton 1969), it implies, combined with the assumption of a constant interest rate, that the investment opportunity is constant. This constant investment opportunity assumption allows us to focus on our main problem abstracting away from other complex issues arising from a stochastic investment opportunity: investigation of effects of default risk of life annuity on optimal consumption and portfolio selection of the retiree and the demand for an annuity product. For treatment of optimal consumption and investment in the face of a stochastic investment opportunity, see Chacko and Viceira (2005), Liu (2007)

[8] In section 5, we extend the model by adding the bequest motive of a retiree, a correlation between stock price and annuity default event, and time-varying mortality rates. We show that our main results are still valid with these extensions.

The retiree receives income at the rate equal to ϵ from life annuity, which she accumulated during her pre-retirement period. We assume that the provider of the annuity is subject to default risk. The retiree can receive part of income from the annuity after the annuity provider's default. Specifically, she recovers her annuity income by the rate of k $(0 \leq k < 1)$ and hence obtains income $k\epsilon$ after the default event. The default event of the annuity provider is driven by an exogenous shock, whose occurrence is distributed according to an exponential distribution with intensity δ; for time $t \geq 0$,

$$\text{Probability of } \{\tau \leq t\} = 1 - e^{-\delta t},$$

where τ stands for the default time of the annuity provider.[9]

We have two sources of risks: stock market risk (a Brownian motion) and insurer's default risk (a Poisson arrival of the default event). The market risk is diversifiable by controlling the investment in the stock, however, the default risk is unhedgeable and cannot be fully diversified away. In this sense, we assume that there is no financial vehicle (securities, financial contracts, or insurance contracts) to hedge against the annuity provider's default risk. For the technical simplicity, we assume that the stock market risk and default risk are independent, i.e., the Brownian motion and Poisson arrival event are assumed to be independent. Accordingly, the financial market including the securities market and the insurance market is considered to be incomplete. Most importantly, the retiree is exposed to an unexpected, exogenous, and permanent reduction in future annuity payments from ϵ to $k\epsilon$ at the insurer's default event.

2.2 The Retiree's Problem in the Presence of Default Risk

We consider the retiree's problem in the presence of insurer default risk. After the default of the annuity provider, we have assumed that the income from annuity will be recovered with the rate of k $(0 \leq k < 1)$. The retiree's problem before the default event is to maximize her CARA lifetime utility (2) by controlling per-period consumption c and risky portfolio π. It

[9]We model the default event as a Poisson jump process. Mortality, disability, retirement, unemployment, and many other events happen at an uncertain time have been considered by such jump process (Merton, 1971; Richard, 1975; Blanchard, 1985; Viceira, 2001). In this paper, the insurer default event also occurs at an uncertain time, so that the default time τ evolving by an exponential distribution can appropriately reflect such uncertain time.

leads to the following value function:

$$V(x) \equiv \max_{(c,\pi)} E\Big[\int_0^\infty e^{-(\beta+\nu+\delta)t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t} - \delta\frac{A}{\gamma}e^{-\gamma(rX_t+k\epsilon)}\Big)dt\Big]. \tag{3}$$

The second term, $-\delta\frac{A}{\gamma}e^{-\gamma(rX_t+k\epsilon)}$, in the right hand side of equation (3) captures the retiree's utility value after the default event, i.e., it is equal to the product of the default intensity δ and the maximized value of her utility after the rate of income from annuity is reduced to $k\epsilon$.[10] Equation (3) shows that the insurer's default risk leads to a new economic problem in which the retiree considers not only her consumption but also wealth at the time of the insurer's default. Notice the presence of the default intensity δ in the second term. For the limiting case of $\delta = 0$, the retiree is not exposed to the default risk and thus she maximizes an objective which is a function solely of intermediate consumption (see e.g., Merton 1969). For the other extreme case where $\delta = +\infty$, the retiree's problem is trivial because the insurer's default is immediate and the rate of income is $k\epsilon$ forever. In this case the problem reduces to the classical optimal consumption and portfolio choice problem (see e.g., Merton 1969).

The wealth process of the retiree satisfies the following dynamics:

$$dX_t = \begin{cases} \big(rX_t - c_t + \epsilon\big)dt + \pi_t\sigma(dW_t + \theta dt), & 0 \le t < \tau \wedge \tau_M, \\ \big(rX_t - c_t + k\epsilon\big)dt + \pi_t\sigma(dW_t + \theta dt), & \tau \wedge \tau_M \le t \le \tau_M, \end{cases}$$

where π is the dollar amount invested in the stock, θ denotes the Sharpe ratio, or $(\mu - r)/\sigma$, and ϵ represents the rate of income from the life annuity. The retiree accumulates wealth at the rates of $(rX - c + \epsilon)$ and $(rX - c + k\epsilon)$ before and after the insurer's default event, respectively. Note that the retiree is exposed to unhedgeable insurer's default risk; the future payments from life annuity decrease from ϵ to $k\epsilon$ at the default event. She is exposed to the stock market risk from stock holdings and hence bears random fluctuations of her wealth, which is captured by the term involving Brownian motion W. Risk taking is compensated by a risk premium and the rate of wealth accumulation is increased by $(\mu - r)\pi$, the product of the risk premium $\mu - r$ and the amount π invested in the stock, relative to the case where the retiree invests only in the risk-free bond.

Because the annuity income is defaultable, it is natural to limit the borrowing with the income. Hence, we impose the nonnegative wealth constraint as the following:

$$X_t \ge 0, \quad 0 \le t \le \tau.$$

[10]For a detailed derivation of equation (3), see an Appendix.

The nonnegative wealth constraint means that the agent cannot borrow money with her future income from the life annuity. See He and Pagés (1993), El Karoui and Jeanblanc-Picqué (1998), Farhi and Panageas (2007), and Dybvig and Liu (2010) for consideration of the constraint. However, we allow for the agent to borrow money with post-default income. This income is insurable due to the provision of minimum subsistence for the incomeless retiree by the society. We limit unsecured borrowing and only allow borrowing for insurable post-default income.

3 Optimal Consumption and Investment Strategies

In this section, we derive the retiree's optimal consumption and investment strategies and discuss their properties.[11] To begin, we state and prove a theorem which expresses the optimal strategies in terms of dual function G and free boundary $\overline{\lambda}$, which are defined in Appendix.

Theorem 3.1 *The retiree's optimal consumption, c^*, and investment in the risky asset, π^*, are given as*

$$\begin{aligned} c_t^* =& rx + \epsilon + \frac{\theta^2}{2\gamma r}\Big(1 + \frac{2}{\theta^2}(\beta + \nu + \delta - r)\Big) - B(\overline{\lambda})\lambda^*(x)^{-\alpha_\delta^*} \\ &+ \frac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)\gamma}\Big[(\alpha_\delta - 1)\lambda^*(x)^{-\alpha_\delta}\int_0^{\lambda^*(x)} \mu^{\alpha_\delta - 2} e^{-\gamma\left(r(G(\mu) - \frac{\epsilon}{r}) + k\epsilon\right)} d\mu \\ &+ (\alpha_\delta^* - 1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2} e^{-\gamma\left(r(G(\mu) - \frac{\epsilon}{r}) + k\epsilon\right)} d\mu\Big], \end{aligned} \tag{4}$$

$$\begin{aligned} \pi_t^* =& \frac{\theta}{\sigma\gamma r} + \frac{\theta}{\sigma}\alpha_\delta^* B(\overline{\lambda})\lambda^*(x)^{-\alpha_\delta^*} + \frac{2\delta A}{\sigma\theta\gamma\lambda^*(x)} e^{-\gamma(rx + k\epsilon)} \\ &- \frac{2\delta A}{\sigma\theta(\alpha_\delta - \alpha_\delta^*)\gamma}\Big[\alpha_\delta(\alpha_\delta - 1)\lambda^*(x)^{-\alpha_\delta}\int_0^{\lambda^*(x)} \mu^{\alpha_\delta - 2} e^{-\gamma\left(r(G(\mu) - \frac{\epsilon}{r}) + k\epsilon\right)} d\mu \\ &+ \alpha_\delta^*(\alpha_\delta^* - 1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2} e^{-\gamma\left(r(G(\mu) - \frac{\epsilon}{r}) + k\epsilon\right)} d\mu\Big], \end{aligned} \tag{5}$$

[11] In an Appendix, the details of deriving optimal strategies for a retiree are provided.

where $\lambda^*(x)$ *is a decreasing function of wealth,* x*, satisfying*

$$\begin{aligned}x+\frac{\epsilon}{r}=&-\frac{1}{\gamma r}\ln\lambda^*(x)-\frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big)+B(\overline{\lambda})\lambda^*(x)^{-\alpha_\delta^*}\\&-\frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}\Big[(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_0^{\lambda^*(x)}\mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu\\&+(\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\overline{\lambda}}\mu^{\alpha_\delta^*-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu\Big],\end{aligned}$$

and

$$\begin{aligned}B(\overline{\lambda})=&\overline{\lambda}^{\alpha_\delta^*}\Big[\frac{\epsilon}{r}+\frac{1}{\gamma r}\ln\overline{\lambda}+\frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big)\\&+\frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}(\alpha_\delta-1)\overline{\lambda}^{-\alpha_\delta}\int_0^{\overline{\lambda}}\mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu\Big].\end{aligned}$$

Here, the two constants of $\alpha_\delta>0$ *and* $\alpha_\delta^*<0$ *are the two roots of the following characteristic equation:*

$$F(\alpha;\delta)\equiv-\frac{1}{2}\theta^2\alpha(\alpha-1)+\alpha(\beta+\nu+\delta-r)+r=0.$$

Note that when $\delta=0$, i.e., there is no default risk, optimal consumption given by equation (19) is equal to the optimal consumption arising from the classical problem in the presence of the nonnegative wealth constraint (Merton 1969, 1971) :

$$c_t=rx+\epsilon+\frac{1}{\gamma r}\Big(\frac{\theta^2}{2}+\beta+\nu-r\Big)-B(\overline{\lambda})\lambda^*(x)^{-\alpha_0^*}. \tag{6}$$

The sum, $rx+\epsilon$, of the first two terms in equation (6) is equal to $r(x+\epsilon/r)$ and corresponds to the part of the retiree's consumption equal to the constant fraction r of her total wealth, defined here as the sum of financial wealth x and the present value ϵ/r of the future income ϵ discounted at the risk-free interest rate. The third term, $\frac{1}{\gamma r}\big(\frac{\theta^2}{2}+\beta+\nu-r\big)$, shows that consumption is higher as the Sharpe ratio θ is higher, the subjective discount rate β higher, the mortality rate ν higher, or the risk aversion coefficient γ lower. The last term shows the effect of the nonnegative wealth constraint. If

$$\frac{\epsilon}{r}+\frac{1}{\gamma r}\ln\overline{\lambda}+\frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big)>0 \tag{7}$$

holds, i.e., $B(\overline{\lambda})>0$, then the last term is negative. This implies the retiree consumes less in the presence of the nonnegative wealth constraint than in its absence. Furthermore, the effect of the constraint on the retiree's consumption becomes larger as her wealth decreases. Intuitively, when facing the constraint, a poor retiree reduces consumption by a larger magnitude than a rich retiree does.

The effect of the default risk on optimal consumption is captured by the extra term in the right-hand side of equation (19); the first term in the bracket is positive while the second is negative. In particular, for a sufficiently large wealth level, the two terms are negligible. Thus, the optimal consumption of a rich retiree in the presence of default risk is not much different from that in its absence. In contrast, if wealth approaches zero, the first term approaches a constant and the second term approaches zero. This implies a poor retiree may consume more in the presence of default risk than in its absence.

Let us now turn to optimal investment in the risky asset. Setting δ to be zero in the right-hand side of (20), we can derive the optimal investment in the absence of default risk:

$$\pi_t = \frac{\theta}{\sigma\gamma r} + \frac{\theta}{\sigma}\alpha_0^* B(\overline{\lambda})\lambda^*(x)^{-\alpha_0^*}. \tag{8}$$

The term $\frac{\theta}{\sigma\gamma r}$ shows that the retiree follows the traditional investment rule; the investment in the stock depends on the Sharpe ration θ and the coefficient γ for absolute risk aversion. As expected, the retiree optimally invests more in the stock as the Sharpe ratio increases or risk aversion decreases.

The nonnegative wealth constraint is reflected in the second term of the right-hand side of (8). Assumption (7) makes the term negative. Similar to consumption, the effect of the constraint on optimal investment is negligible when wealth is large enough, and becomes more significant when her wealth gets smaller. That is, a poor retiree has relatively small stock holdings due to the constraint, whereas a rich retiree is little influenced by it. The effect of the default risk on optimal investment is reflected in the third term and terms involving the bracket of the right-hand side of equation (20). The third term is positive, and consequently, the retiree tends to take a riskier position due to the default risk. However, the terms involving the bracket, in total, are negative. Combining all the effects, the adjustment of investment in the stock due to the default risk is ambiguous, i.e., the retiree's optimal investment in the stock can be either larger or smaller in the presence of default risk than in its absence. When wealth is sufficiently large, the optimal investment strategy in the presence of default risk is not much different from that in its absence; a sufficiently rich retiree is little affected by the default risk in formulating her investment.

4 Implications

In this section, we explain optimal strategies by using numerical solutions. The parameters for the baseline case are set as follows: $r = 3.71\%$, the annual rate of return from rolling-over of 1-month T-bills during the time period of 1926-2009,[12] $\mu = 11.23\%$ and $\sigma = 19.54\%$, the return and standard deviation of a portfolio consisting of the world's large stocks during the same time period.[13] We also assume $\beta = r$ and set $\gamma = 2$. We take the annual rate of annuity to be 1, i.e., $\epsilon = 1$, and the retiree's expected lifetime to be 20 years, or equivalently, $\nu = 0.05$.

Default intensity δ is calibrated based on Moody's (2012) historical data of average cumulative issuer-weighted global default rates by rating categories for the time period from 1983 to 2011. Similarly, Lopes and Michaelides (2007) and Babbel and Merrill (2007) use corporate debt ratings by a credit rating agent like Moody's to estimate default probability of life annuities. The statistics related to federal, state, municipal and other local governments are not available in sufficiently credible quantities, so Moody's historical data on corporate defaults is used to calibrate the insurer default intensity, δ. The deterioration of corporate credit rating constitutes major risks to life insurers (Impavido and Tower, 2009). We select four categories, Aaa, Aa, A, and B, to calibrate the default intensity. It is calibrated as follows: 0.0001 for Aaa, 0.0012 for Aa, 0.0030 for A, 0.0526 for B.

In the U.S. the NOLHGA and its state government members provide retired people with minimum benefit guarantees by recovering parts of remaining benefits after default of annuity providers. The coverage limit for most of states is $\$100,000$ in withdrawal and cash values for annuities. As a result, the rest of the coverage exceeding the coverage limit is recovered according to the liquidation ratio, which is the amount of assets available in the insolvency estate. In light of this fact, we regard the recovery rate as the recovery amount of the excess claim. We take the three values of k; following Babbel and Merrill (2007), we set the recovery rate, k, of the excess claim to be 0%, 10%, and 25%.

4.1 Optimal Consumption and Investment in the Risky Asset

Figure 2 displays the retiree's optimal consumption and investment in the risky asset as a function of initial wealth for different levels of default risk when there is no recovery after

[12]Source: Bureau of Labor Statistics.

[13]pp. 170 of Bodie, Kane, and Marcus (2011).

annuity provider defaults ($k = 0$). As expected, optimal consumption increases as wealth increases, and decreases as default risk increases. Figure 2 shows a pattern for the investment in the risky asset quite different than has been found in the literature. According to Bodie *et al.* (1992), agents with a higher income risk tend to take lower risk when making investment in financial assets. However, in this paper we find that higher default risk may lead retired people to increase their stock holdings.[14] This is because the defaultable annuity is no longer a safe asset, and, thus, the retiree is willing to take a much larger investment in the stock to absorb the default risk.

[Insert Figure 2 here.]

Changes in Recovery Rate

Table 1 shows optimal consumption and investment in the risky asset for several recovery rates and default intensities. As the life annuity has a higher recovery rate or the annuity provider has a higher credit rating, the retiree consumes more and invests less in the risky asset. This is because a higher recovery rate of the life annuity and a higher credit rating of the annuity provider can mitigate the impact of default event of the life annuity on the retiree's optimal behavior. Also the table shows that a drastic change in the credit rating of the annuity provider has a significant effect on the retiree's optimal behavior even if the recovery rate is substantially large. For instance, optimal consumption decreases from 2.6778 to 2.3641 and optimal investment in the risky asset increases from 17.8819 to 22.6893 if the credit rating of the annuity provider changes from A to B, for the case where $k = 0.25$ and $x = 20$.

[Insert Table 1 here.]

Changes in Risk Aversion

Figure 3 shows the effects of changes in risk aversion on the retiree's optimal behavior. The figure demonstrates that lower risk aversion leads the retiree to take larger consumption

[14] In our model the retiree has an exponential utility function and exhibits neither decreasing (absolute) prudence nor decreasing (absolute) temperament, so the agent may take more risk in the face of higher background risk (Kimball, 1993). The background risk is an undiversifiable risk affecting an economic agent's behavior. Typical examples of background risk are income risk and risk of house ownership. See Bodie *et al.* (1992), Koo (1998), and Heaton and Lucas (2000) for study of income risk as background risk.

and larger investment in the risky asset. The figure also shows that the retiree increases investment in the risky asset substantially in the face of higher default risk and the increase is larger when she has lower risk aversion. It implies that the stock investment can be used as an imperfect hedging tool against the default risk.

[Insert Figure 3 here.]

Changes in Investment Opportunity

Tables 2 and 3 show the relationship between the investment opportunity and the retiree's optimal choice. For simplicity we assume that the recovery rate is 0. As expected, the two tables show that the retiree is willing to consume more and invest a larger amount in the risky asset if the investment opportunity is better, or equivalently, if the expected return on the risky asset is higher and/or the volatility of the return is lower.

[Insert Table 2 and 3 here.]

Changes in Default Risk

We see from Table 2 that the retiree might drastically reduce her consumption in the face of a high default risk. For example, Table 2 shows that consumption declines from 2.7732 (2.5103) to 2.4339 (1.9599) when the retiree has initial wealth 20, the credit rating of the insurance company changes from A to B, and the expected rate of stock return is equal to 0.1223 (0.1023, respectively). Reduction in consumption due to a higher default risk is more significant if the investment opportunity is worse.

On the other hand, the investment in the risky asset increases as the the retiree is exposed to a higher default risk of the life annuity. We have already shown that the risky asset may be a good substitute for the defaultable annuity and the retiree increases her investment in the risky asst in order to absorb the default risk. For instance, the retiree with initial wealth of 30 increases her investment in the risky asset from 22.8493 (19.5456) to 28.3619 (23.2016) as the credit rating of the annuity provider declines from A to B if the volatility of the risky asset is 0.1854 (0.2054, respectively). Apparently a better investment opportunity makes the positive effect of default risk on the optimal investment in the risky asset more pronounced.

4.2 Annuity Demand and Insurer Default Risk

4.2.1 Implicit Value of Life Annuity

In this section, we calculate the actuarially-fair price and the retiree's implicit value of life annuity and compare the two values. The implicit value of life annuity is the marginal rate of substitution between annuity holdings and financial wealth (Koo, 1998), so it is the retiree's subjective marginal value of her annuity holdings, i.e., her reservation price of an annuity when she is offered to purchase or sell a small amount of the annuity. It is a proxy for the retiree's annuity demand; a higher value than the market price implies that the retiree would be willing to buy it and a lower value than the market price implies the opposite, i.e., she would be willing to sell it if she were given an opportunity to do so.

The actuarially-fair price serves as a benchmark to set the market price by the annuity provider (Mitchell *et al.*, 1999). The actuarially-fair price can be calculated as

$$E\Big[\int_0^{\tau\wedge\tau_M} \epsilon e^{-rt}dt\Big] = \frac{\nu + k\delta}{(r+\nu+\delta)(\nu+\delta)}\epsilon,$$

and, for comparison purposes, we define the *benchmark price* as the actuarially-fair price without the default risk ($\delta = 0$), $\dfrac{\epsilon}{r+\nu}$. We introduce the definition of the implicit value of life annuity.

Definition 4.1 *Let us denote the value function defined in (3) by $V(x, \epsilon, \delta)$, which is a function of initial wealth x, rate ϵ of income from the annuity, and default intensity δ. Then the implicit value of life annuity is defined by*

$$\frac{\partial V(x,\epsilon,\delta)}{\partial \epsilon}\Big/\frac{\partial V(x,\epsilon,\delta)}{\partial x}.$$

Figure 4 depicts the implicit value of life annuity as a function of wealth. It shows that the implicit values increase as wealth level increases if the insurance provider has a high credit rating of Aaa or Aa. It implies that the retiree, who is not much exposed to default risk of the life annuity, is increasingly interested in taking up more of the life annuity as her wealth gets larger. This is consistent with the finding of Inkmann *et al.* (2010), who claim that annuities are attractive to rich retirees. However, if the default risk of life annuity is significantly high (e.g., see the case of B), the implicit value is substantially smaller than the benchmark price. This implies that lower default risk can substantially increase the demand for annuities.

[Insert Figure 4 here.]

Changes in Recovery Rate

In Figure 5, we show how the implicit value of life annuity varies with the recovery rate. We see that the implicit value increases with the recovery rate. Notice that the implicit value of the life annuity is more significantly affected by changes in the recovery rate if the credit rating of the insurance provider is lower. Intuitively, the retiree is less concerned about default of the annuity provider whose the credit rating is good, and subsequently, the recovery rate hardly affects the implicit value of the annuity. However, when the retiree is exposed to a higher default risk, a larger recovery leads to a higher implicit value because the recovery plays a crucial role in mitigating the default risk.

[Insert Figure 5 here.]

Changes in Risk Aversion

Lopes and Michaelides (2007) have studied a model with defaultable annuities and conclude that the significantly low annuity take-up cannot be explained by the possibility of rare events such as default of the annuity provider. They have asserted that it would be necessary to assume high risk aversion for a rare event to change behavior of a retiree, but high risk aversion makes annuities more attractive as insurance against mortality risk, as a result, a rare event would unlikely provide the main explanation for the low annuity demand. Babbel and Merrill (2007) have also made a contribution to the literature by finding that the participation in the annuity market can be low when retirees are exposed to default risk of annuity providers. In this paper we provide a reconciliation of the seemingly contradictory conclusions reached by the two teams of researchers. We show that the retiree's annuity demand increases as her risk aversion increases if default risk is small, consistent with Lopes and Michaelides' finding. We also show that the demand decreases with risk aversion if default risk is large, consistent with Babbel and Merrill's result.

Figure 6 reveals that higher risk aversion induces retirees to purchase larger amounts of annuities for the moderate default risk of annuity provider (e.g., see the cases of Aaa and Aa). Most importantly, the effect of risk aversion on the annuity take-up is not monotonic. In the figure, we find that when the default risk is substantial (e.g., see the case of B), the annuity demand decreases as risk aversion increases.

[Insert Figure 6 here.]

Changes in Investment Opportunity

Table 4 shows how the implicit value of the life annuity changes with investment opportunity. When the default risk is low, the implicit value falls if the investment opportunity gets better, consistent with findings in Previtero (2010) and Chalmers and Reuter (2012). However, if the retiree is exposed to relatively higher default risk (e.g., for the cases of Aa, A, and B), our model can produce an opposite result to theirs for some wealth levels; the life annuity becomes less valuable in the presence of a high default risk as the investment opportunity worsens. That is, the defaultable annuity may not be a good substitute for the risky stock and, when the stock becomes a less favorable investment vehicle, the life annuity becomes less favorable too.

[Insert Table 4 here.]

4.2.2 Certainty Equivalent Wealth Gain

In this section, we measure the impacts of insurer default risk by introducing a concept of certainty equivalent wealth gain (CEWG). We define the CEWG as the largest wealth that the retiree is willing to give up in exchange for making the annuity default-free. That is, the CEWG is a kind of compensation for the retiree in return for bearing default risk of life annuity. We discuss its various properties for changes in parameter values.

Definition 4.2 $\Delta(x)$ *is called the certainty equivalent wealth gain at wealth level* x *if it satisfies*

$$V(x - \Delta(x), \epsilon, 0) = V(x, \epsilon, \delta).$$

Figure 7 shows that the CEWG is a decreasing function of wealth, since a retiree will be more able to absorb default risk as she gets richer.[15] Moreover, the higher the default risk, the larger the CEWG. Since a high default risk gives rise to a large negative expected wealth shock to the retiree, as a result, she requires a large CEWG.

[15] The CEWG converges to zero as wealth approaches infinity. A retiree who is exposed to the default risk cannot expect to receive annuity income over her entire lifetime, so she requires a positive CEWG unless her wealth is infinity.

[Insert Figure 7 here.]

Changes in Recovery Rate

The sensitivity of the CEWG to changes in the recovery rate is illustrated in Figure 8. As expected, a higher recovery rate induces the retiree to require a smaller compensation for default risk of the life annuity.

[Insert Figure 8 here.]

Changes in Risk Aversion

Figure 9 shows that the CEWG increases with the degree of a retiree's risk-averse. Obviously, the retiree with larger risk aversion requires a larger compensation for default risk of the life annuity. Moreover, the change in the CEWG with risk aversion is more significant if default risk is larger (B versus Aaa, Aa, or A).

[Insert Figure 9 here.]

Changes in Investment Opportunity

Table 5 exhibits sensitivities of the CEWG to changes in expected return μ and volatility σ of the risky asset. The retiree has a tendency to require a larger CEWG when the expected return on the stock is lower and/or its volatility is higher. Intuitively, the retiree requires a higher risk premium for default risk of the life annuity as the investment opportunity worsens.

[Insert Table 5 here.]

5 Robustness

We have shown by numerical examples that high default risk leads a retiree to reduce her consumption and increase stock holdings and reduce life annuity holdings. We have assumed that the retiree has no bequest motive, the mortality rate, ν, is constant, and the correlation between the risky asset price and the default event of the insurance provider is zero. In this section, we will show that the conclusion is robust to changes in the assumption.

We first consider the bequest motive of a retiree. The bequest motive can decrease the annuity take-up for the retiree. This is because a retired person would like to bequeath part of wealth to heirs at death. Actually, Friedman and Warshawsky (1990), Brown (2001), and Johnson *et al.* (2004) demonstrate that the strong bequest motive of annuity buyers can generate low annuity demand. Therefore, addition of the bequest motive will only reinforce our conclusion. We confirm that our results are robust to allowing the bequest motive.

For the next, we consider a correlation between the risky asset price and default intensity and examine its impact on the annuity demand (or equivalently, the implicit value of life annuity). Throughout this section, we assume that the risky asset considered in our model is a well-diversified portfolio such as a worldwide stock index, thus, the price fluctuation of the risky asset is solely dependent on a systematic risk factor. We relate the systematic risk factor to the default risk of annuity provider by imposing correlation between them. Specifically, we employ process δ_s introduced by Blanchet-Scalliet *et al.* (2008) as default intensity;[16] for time $t \geq 0$,

$$\text{Probability of } \{\tau \leq t\} = \int_0^t \delta_s ds,$$

with δ_s satisfying

$$d\delta_s = a\delta_s ds + b\delta_s dW_s^*, \quad \delta_0 = \delta > 0, \ a < 0, \ b > 0,$$

where W_s^* is a standard Browninan motion such that

$$dW_s \cdot dW_s^* = \rho ds, \quad \text{for} \quad \rho \in [-1, 1].$$

Furthermore, we will consider the case that the probability of death of the retiree changes over time. We consider the following mortality rate, ν_t; for time $t \geq 0$,

$$\text{Probability of } \{\tau_M \leq t\} = \int_0^t \nu_s ds,$$

with ν_s satisfying

$$d\nu_s = \tilde{a}\nu_s ds + \tilde{b}\nu_s d\widetilde{W}_s, \quad \nu_0 = \nu > 0, \ \tilde{a} < 0, \ \tilde{b} > 0,$$

where $\widetilde{W}_s$ is a standard Browninan motion independent of W_s and W_s^*.[17]

The detailed procedure to get a solution under this situation is explained in an Appendix.

[16]The process, δ_s, of Blanchet-Scalliet *et al.* (2008) does not satisfy $\int_0^\infty \delta_s ds \leq 1$. Nonetheless, they utilize the default probability to get a simple solution. The assumption leads us to get useful economic intuition without technical complications.

[17]Letting $a = -\delta$ ($\tilde{a} = -\nu$), and $b = 0$ ($\tilde{b} = 0$), the time of default, τ, (the time of death, τ_M) follows an exponential distribution with intensity δ (ν, respectively). Therefore, the model in the previous sections can be considered as a special case of the model in this section.

5.1 Bequest Motive of a Retiree

In Table 6, the parameter $\tilde{k}$ denotes a weight for bequest motive of a retiree; the bequest motive increases with respect to an increase of $\tilde{k}$. From the table, we can confirm the intuition of Friedman and Warshawsky (1990), Brown (2001), and Johnson *et al.* (2004) that the annuity demand of the retiree decreases as the bequest motive increases. Furthermore, regardless of the bequest motive, the implicit value of life annuity decreases as the credit rate of annuity provider worsens. Importantly, we find that the increase of bequest motive reinforces the negative effect of insurer default risk on the annuity demand; the negative shock from an unexpected and exogenous decrease in future annuity income might induce the retiree to lower the annuity take-up; she tries to bequeath larger wealth to her heirs.

[Insert Table 6 here.]

5.2 Correlation between Stock Price and Annuity Default Event

Table 7 shows how implicit values of A and B-rated life annuities vary according to changes in the volatility of default intensity, b, and the correlation, ρ, between the risky asset price and the default intensity of life annuity. In the table, the implicit values of life annuities are higher for the case $\rho = 10\%$, i.e., correlation is positive than for the case $\rho = -10\%$, i.e., correlation is negative. This is because life annuities are hedging tools against the market risk if the correlation is positive, so the implicit values are higher.

The table shows, however, that our basic conclusion is still valid, i.e., high default risk leads the retiree to reduce her consumption and increase stock holdings and reduce life-annuity holdings.

[Insert Table 7 here.]

5.3 Time-Varying Mortality Rates

Table 8 shows that, as risk in mortality rate, $\tilde{b}$, increases, the implicit value of life annuity decreases for a good credit rating such as Aaa, while it increases for a poor credit rating such as B. In our model with negative exponential utility, the implicit value of life annuity income in the absence of default risk declines as uncertainty in death time increases. Thus, for a good credit rating the increase in marginal utility of life annuity due to increase in mortality risk declines as $\tilde{b}$ increases. For a poor credit rating, the annuity has an option-like feature and its implicit value increases as $\tilde{b}$ increases. The table, however, still supports that our basic conclusion is valid.

[Insert Table 8 here.]

6 Conclusion

We investigate the optimal consumption and portfolio choice problem for a retiree who partially annuitized her wealth in the presence of default risk of life annuity. The default risk stems from a default event of the annuity provider, which can be neither hedged nor insured. The financial market considered in this paper is thus incomplete. We develop a new dynamic programming approach to solve the optimal consumption and portfolio choice problem in such incomplete market.

We introduce two useful concepts to investigate the impact of default risk of life annuity on optimal consumption and investments of retirees; implicit value of annuity, and certainty equivalent wealth gain. By utilizing them we show that a retiree may have a strong motive for selling or refunding her annuities when facing high default risk. When the default risk is substantial, the annuity demand decreases as risk aversion increases. Furthermore, the life annuity becomes less valuable in the presence of a high default risk as the investment opportunity worsens. Hence, the annuity demand is significantly affected by insurer default risk.

An interesting extension of this paper is to endogenize annuitization for a retiree. If we consider an optimal choice of an annuity, we can determine its equilibrium price and investigate the relationship between the price and its default risk.

7 Appendix

7.1 The Details of State Guaranty Associations

A number of insurance companies have experienced substantial losses in economic downturns and a few of them even filed for bankruptcy. In the U.S. insurance companies such as Executive Life Insurance Co., First Capital Life Insurance Co., Monarch Life Insurance Co. failed after

1990 (Chen and Suchanecki, 2007), and the AIG Group underwent a liquidity crisis on 2008 and received a substantial government bailout of $85 billion, which is the largest government bailout of a private company. In the United Kingdom National Assurance Co., English and American Insurance Co., Oaklife Assurance Co., and Equitable Life, the world's oldest life insurer, became insolvent in 1980s and 1990s. In the Netherlands, the ING Group received a 10 billion euro capital injection and Aegon a billion euro recapitalization fund during the 2008 global crisis.

However, in the U.S. and many other jurisdictions there are guarantee funds including state guaranty associations, so that individuals are possible to diversify default risks of insurance companies. While the maximum coverage limits vary from state to state, most states offer $100,000 coverage in withdrawal and cash values for annuities. Actually, in case the claim of a policyholder exceeds the guaranty association coverage, the recovery amount of her excess claim depends highly on the liquidation ratio for the insolvency.

> It is a common misunderstanding that policyholder recoveries in insurance liquidations are limited to guaranty association coverage limits or "caps." The truth is that whether a policyholder recovers all or most of her claim *above* guaranty association caps depends significantly on whether regulatory intervention occurs before the failed company's assets have been substantially dissipated, and whether assets are effectively protected and marshaled in the company's receivership (Source: NOLHGA, 2011).

As the excerpt from the NOLHGA (2011) demonstrates, the recovery amount of the claim exceeding guaranty association limits depends crucially on the liquidation ratio for the insolvency of insurance company. Consider a policyholder who has a claim of $1 million. Firstly, we suppose that there is no guaranty association protection. When the life annuity defaults and the liquidation ratio is 95%, the policyholder recovers $950,000 of the $1 million claim. However, if the liquidation ratio is zero, then the policyholder recovers nothing (Figure 10).

For the next, we consider a case of guaranty association limit is $100,000. Of course, the policyholder recovers all of the claim within $100,000 when insurance company defaults. However, because she has a claim of $1 millon, the recovery amount of the excess claim is determined according to the liquidation ratio. If we assume that the liquidation ratio is 95%, then the policyholder recovers \$855, 000 of the excess claim of \$900, 000. Hence, the amount of total recovery becomes \$955, 000 (\$100, 000 plus \$855, 000). However, when the liquidation ratio is 0%, then the excess claim is not recovered at all (Figure 10).

Finally, we suppose that the guaranty association cap is \$250, 000. If the liquidation ratio is 95%, then the recovery amount of the excess claim for the policyholder is \$712, 500. As a result, the total recovery for the policyholder is \$962, 000 (\$250, 000 plus \$712, 500). Note that even if the guaranty association cap is 2.5 times larger as we compared to the previous \$100, 000, the total recovery increases a little, which is \$7, 500. Furthermore, if the liquidation ratio is zero, then the policyholder bears a very large loss of \$750, 000 (Figure 10). Accordingly, the excess claim is recovered by not the level of guaranty association limit, but the liquidation ratio.

According to the recent 2012 report of Moody's, the average "firm-wide" liquidation ratio for the 10 default resolutions was 53.4% in 2011. Here, the firm-wide rate represents the weighted-average recovery rate across all of the issuer's debts in which the weights are the size of the debts. As we compared to 63.0% for 64 companies defaulted in 2010, the lower liquidation rate of 53.4% is induced by the ultimate resolutions for more than 1,000 default events in 2011. Hence, in the insolvencies claims on life annuities have been paid at a significant level.

7.2 Details of Deriving Solutions

A retiree receives life annuity with the rate of $k\epsilon$ after insurer default event. Then post-default problem is exactly same as classical Merton's (1969, 1971) problem. Specifically, the value function follows

$$J(x) \equiv \max_{(c,\pi)} E\Big[\int_0^\infty e^{-(\beta+\nu)t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt\Big],$$

which is subject to the wealth process

$$dX_t = (rX_t - c_t + k\epsilon)dt + \pi_t\sigma(dW_t + \theta dt), \quad t \geq 0.$$

The optimality conditions for consumption c and risky investment π are

$$c_t^* = J'(x)^{-1/\gamma} \quad \text{and} \quad \pi_t^* = -\frac{\mu - r}{\sigma^2}\frac{J'(x)}{J''(x)}.$$

The dynamic programming principle allows us to derive the following Hamilton-Jacobi-Bellman (HJB) equation:

$$(rx + k\epsilon)J'(x) - \frac{1}{2}\theta^2 \frac{J'(x)^2}{J''(x)} - (\beta + \nu)J(x) + \frac{1}{\gamma}J'(x)\{\ln J'(x) - 1\} = 0.$$

Following Merton (1969, 1971), we obtain a solution to the HJB equation

$$J(x) = -\frac{A}{\gamma}e^{-\gamma(rx+k\epsilon)} \quad \text{for} \quad A = \frac{1}{r}e^{-\frac{1}{r}(\frac{\theta^2}{2}+\beta+\nu-r)}.$$

Moreover, the optimal consumption and investment strategies can be described as

$$c^M = rx + k\epsilon + \frac{1}{\gamma r}\Big(\frac{\theta^2}{2} + \beta + \nu - r\Big) \quad \text{and} \quad \pi^M = \frac{\theta}{\gamma r \sigma}.$$

Notice that the optimal consumption is a linear function of wealth x and income $k\epsilon$, and the optimal investment in the risky asset (stock) is constant.

The retiree's problem in the presence of insurer default risk is

$$V(x) = \max_{(c,\pi)} E\Big[\int_0^\infty e^{-(\beta+\nu+\delta)t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t} - \delta\frac{A}{\gamma}e^{-\gamma(rX_t+k\epsilon)}\Big)dt\Big],$$

which is subject to the wealth process

$$dX_t = (rX_t - c_t + \epsilon)dt + \pi_t\sigma(dW_t + \theta dt), \quad t \geq 0.$$

Here, the term $-\delta\frac{A}{\gamma}e^{-\gamma(rX_t+k\epsilon)}$ of the retiree's problem reflects the value function after insurer default event, which is adjusted by the possibility δ of insurer default risk. The optimality conditions for consumption c and risky investment π are given by

$$c_t^* = V'(x)^{-1/\gamma} \quad \text{and} \quad \pi_t^* = -\frac{\mu - r}{\sigma^2}\frac{V'(x)}{V''(x)}. \tag{9}$$

Then the HJB equation can be stated as

$$\begin{aligned} -(\beta + \nu + \delta)V(x) + (rx + \epsilon)V'(x) - \frac{1}{2}\theta^2\frac{V'(x)^2}{V''(x)} \\ + \frac{1}{\gamma}V'(x)\{\ln V'(x) - 1\} - \frac{A\delta}{\gamma}e^{-\gamma(rx+k\epsilon)} = 0. \end{aligned} \tag{10}$$

As we mentioned previously, the financial market is incomplete, since default risk of life annuity cannot be diversified away. A consumption and portfolio selection problem in an incomplete market is in general difficult to solve. We develop a dynamic programming approach by using a *modified* dual function, called G, of the original value function V. Differentiating both sides of (10) with respect to x yields

$$-(\beta+\nu+\delta)V'(x)+rV'(x)+(rx+\epsilon)V''(x)-\frac{1}{2}\theta^2\frac{2V'(x)V''(x)^2-V'(x)^2V'''(x)}{V''(x)^2} + \frac{1}{\gamma}V''(x)\ln V'(x)+Ar\delta e^{-\gamma(rx+k\epsilon)}=0. \tag{11}$$

If we define

$$\lambda(x)\equiv V'(x)$$

and introduce the dual function G,

$$G\big(\lambda(x)\big)\equiv x+\frac{\epsilon}{r}, \tag{12}$$

then G is a decreasing function and satisfies

$$G'\big(\lambda(x)\big)\lambda'(x)=1 \quad \text{and}$$
$$G''\big(\lambda(x)\big)\lambda'(x)^2+G'\big(\lambda(x)\big)\lambda''(x)=0$$

for $0<\lambda<\overline{\lambda}$, where $\overline{\lambda}$ corresponds to the zero wealth level. Then

$$G(\overline{\lambda})=\frac{\epsilon}{r} \quad \text{and} \quad G^{'}(\overline{\lambda})=0, \tag{13}$$

where the second equation follows from the zero risky investment when wealth approaches zero (Farhi and Panageas, 2007; Dybvig and Liu, 2010). We can now rewrite HJB equation (11) as

$$-\frac{1}{2}\theta^2\lambda^2G''(\lambda)-\lambda G'(\lambda)\{\theta^2+\beta+\nu+\delta-r\}+rG(\lambda)+Ar\delta e^{-\gamma\left(r(G(\lambda)-\frac{\epsilon}{r})+k\epsilon\right)}G'(\lambda)=-\frac{1}{\gamma}\ln\lambda. \tag{14}$$

We conjecture a solution of the following form to equation (14):

$$G(\lambda)=-\frac{1}{\gamma r}\ln\lambda-\frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big)+\eta(\lambda)\lambda^{-\alpha_\delta}+\eta^*(\lambda)\lambda^{-\alpha^*_\delta}, \quad \text{subject to } \eta'(\lambda)\lambda^{-\alpha_\delta}+(\eta^*(\lambda))'\lambda^{-\alpha^*_\delta}=0, \tag{15}$$

where $\alpha_\delta>0$ and $\alpha^*_\delta<0$ are the two roots of the following characteristic equation

$$F(\alpha;\delta)\equiv-\frac{1}{2}\theta^2\alpha(\alpha-1)+\alpha(\beta+\nu+\delta-r)+r=0.$$

Plugging (15) into (14), we get the following implicit equation for the dual function:

$$
\begin{aligned}
G(\lambda) =& -\frac{1}{\gamma r}\ln\lambda - \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big) + B(\overline{\lambda})\lambda^{-\alpha_\delta^*} \\
&- \frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}\Big[(\alpha_\delta-1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu \\
&+ (\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^*-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu\Big],
\end{aligned}
\tag{16}
$$

where

$$
B(\overline{\lambda}) \equiv \eta(\overline{\lambda}) + \frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}\overline{\lambda}^{\alpha_\delta^*-1}.
$$

Furthermore, equations in (13) imply the following two relationships:

$$
\begin{aligned}
\frac{\epsilon}{r} =& -\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big) + B(\overline{\lambda})\overline{\lambda}^{\,-\alpha_\delta^*} \\
&- \frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}(\alpha_\delta-1)\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu,
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
\frac{1}{\gamma r} + \frac{\alpha_\delta^*\epsilon}{r} + \frac{\theta^2\alpha_\delta^*}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big) =& -\alpha_\delta^*\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{2\delta A}{\theta^2\gamma}\frac{1}{\overline{\lambda}} \\
&+ \frac{2\delta A(\alpha_\delta-1)}{\theta^2\gamma}\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu.
\end{aligned}
\tag{18}
$$

Now, it is ready to obtain optimal consumption and risky investment strategies in the presence of insurer default risk. Optimality conditions given by (9) are rewritten by using the dual function G. Specifically,

$$
c_t^* = \lambda^*(x)^{-1/\gamma} \quad \text{and} \quad \pi_t^* = -\frac{\mu-r}{\sigma^2}\lambda^*(x)G'\big(\lambda^*(x)\big).
$$

By utilizing the definition (12) of dual function G and the general solution (16), we obtain the following optimal strategies:

$$
\begin{aligned}
c_t^* =& rx + \epsilon + \frac{\theta^2}{2\gamma r}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big) - B(\overline{\lambda})\lambda^*(x)^{-\alpha_\delta^*} \\
&+ \frac{2\delta A r}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}\Big[(\alpha_\delta-1)\lambda^*(x)^{-\alpha_\delta}\int_0^{\lambda^*(x)} \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu \\
&+ (\alpha_\delta^*-1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\overline{\lambda}} \mu^{\alpha_\delta^*-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu\Big],
\end{aligned}
\tag{19}
$$

$$\pi_t^* = \frac{\theta}{\sigma\gamma r} + \frac{\theta}{\sigma}\alpha_\delta^* B(\overline{\lambda})\lambda^*(x)^{-\alpha_\delta^*} + \frac{2\delta A}{\sigma\theta\gamma\lambda^*(x)} e^{-\gamma(rx+k\epsilon)}$$
$$- \frac{2\delta A}{\sigma\theta(\alpha_\delta - \alpha_\delta^*)\gamma}\Big[\alpha_\delta(\alpha_\delta - 1)\lambda^*(x)^{-\alpha_\delta}\int_0^{\lambda^*(x)} \mu^{\alpha_\delta - 2} e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)} d\mu \tag{20}$$
$$+ \alpha_\delta^*(\alpha_\delta^* - 1)\lambda^*(x)^{-\alpha_\delta^*}\int_{\lambda^*(x)}^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2} e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)} d\mu\Big].$$

Note that if we apply the iterative numerical algorithm given in Appendix of the paper of How Annuity Demand is Affected by Insurer Default Risk, then we can determine $\overline{\lambda}$ and $B(\overline{\lambda})$ and subsequently, demonstrate implications by using numerical solutions.

In the next section, we provide the uniqueness and existence of the solution $\overline{\lambda}$ of (18) and the monotonic decreasing property of G.

7.3 Theorems and Proofs Concerning the Uniqueness, Existence, and Monotonicity

Theorem 7.1 *(Uniqueness) If* $\dfrac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}} < 1$, *then the solution of (16) is unique.*

Proof. Let G_1 and G_2 be the solutions of (16). Then we can obtain

$$G_1(\lambda) - G_2(\lambda) = -\frac{2\delta A}{\theta^2(\alpha_\delta - \alpha_\delta^*)\gamma}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_0^{\lambda} \mu^{\alpha_\delta - 2}\Big(e^{-\gamma\left(r(G_1(\mu)-\frac{\epsilon}{r})+k\epsilon\right)} - e^{-\gamma\left(r(G_2(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}\Big)d\mu$$
$$+(\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2}\Big(e^{-\gamma\left(r(G_1(\mu)-\frac{\epsilon}{r})+k\epsilon\right)} - e^{-\gamma\left(r(G_2(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}\Big)d\mu\Big].$$

The inequality of

$$\left|e^{-\gamma\left(r(G_1(\mu)-\frac{\epsilon}{r})+k\epsilon\right)} - e^{-\gamma\left(r(G_2(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}\right| < \gamma r|G_1(\mu) - G_2(\mu)|$$

gives

$$|G_1(\lambda) - G_2(\lambda)| \le \frac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}}\Big(\frac{\lambda}{\overline{\lambda}}\Big)^{-\alpha_\delta^*}\sup_{\mu}|G_1(\mu) - G_2(\mu)|.$$

Hence, for $0 < \lambda < \overline{\lambda}$, when $\dfrac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}} < 1$ the solution of (16) is unique. **Q.E.D.**

Theorem 7.2 *(Monotonicity) Suppose* $0 \le x \le x^*$ *for a positive and finite constant* x^* *and* $0 < \overline{\lambda} \le 1$. *If*

$$\frac{1}{\gamma r} + \alpha_\delta^*\Big\{\frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta + \nu + \delta - r)\Big)\Big\} > 0 \tag{21}$$

holds, then a solution of (14) satisfies $G'(\lambda) < 0$.

Proof. A solution of (14) satisfies the integral equation (16), thus, we can obtain

$$
\begin{aligned}
G'(\lambda) \quad &= -\frac{1}{\gamma r}\frac{1}{\lambda} - \alpha_\delta^* B(\overline{\lambda})\lambda^{-\alpha_\delta^*-1} - \frac{2\delta A}{\theta^2\gamma}\frac{1}{\lambda^2}e^{-\gamma\left(r(G(\lambda)-\frac{\epsilon}{r})+k\epsilon\right)} \\
&+ \frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}\Big[\alpha_\delta(\alpha_\delta-1)\lambda^{-\alpha_\delta-1}\int_0^\lambda \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu \\
&+ \alpha_\delta^*(\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*-1}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^*-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\le -\frac{1}{\gamma r}\frac{1}{\lambda} - \alpha_\delta^* B(\overline{\lambda})\lambda^{-\alpha_\delta^*-1} + \frac{2\delta A}{\theta^2\gamma}\frac{1}{\lambda^2}\Big(e^{-\gamma(rx^*+k\epsilon)} - e^{-\gamma(rx+k\epsilon)}\Big) + \frac{2\delta A\alpha_\delta^*}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}\lambda^{-\alpha_\delta^*-1}\overline{\lambda}^{\alpha_\delta^*-1} \\
&\le -\frac{1}{\lambda}\Big[\frac{1}{\gamma r}(1+\alpha_\delta^*\ln\overline{\lambda}) + \alpha_\delta^*\Big\{\frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big)\Big\}\Big] \\
&\le -\frac{1}{\lambda}\Big[\frac{1}{\gamma r} + \alpha_\delta^*\Big\{\frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big)\Big\}\Big] \\
&< 0.
\end{aligned}
$$

The third inequality is derived by putting $B(\overline{\lambda})$ into (17) and using the assumption of $0 \le x \le x^*$, and the fourth inequality is due to the assumption of $0 < \overline{\lambda} \le 1$. Since

$$
\frac{1}{\gamma r} + \alpha_\delta^*\Big\{\frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big)\Big\} > 0,
$$

obviously the last inequality holds. **Q.E.D.**

We find lower and upper bounds for the free boundary $\overline{\lambda}$ of our problem.

Proposition 7.1 *Let*

$$
\begin{aligned}
L_\delta &\equiv \frac{1}{\gamma r} + \alpha_\delta^*\Big\{\frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big)\Big\}, \\
\psi_\delta(\overline{\lambda}) &\equiv -\alpha_\delta^*\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{2\delta A}{\theta^2\gamma}\frac{1}{\overline{\lambda}} + \frac{2\delta A(\alpha_\delta-1)}{\theta^2\gamma}\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu, \\
\underline{\phi}_\delta(\overline{\lambda}) &\equiv -\alpha_\delta^*\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{2\delta A}{\theta^2\gamma}\frac{1}{\overline{\lambda}}, \ \textit{and} \\
\overline{\phi}_\delta(\overline{\lambda}) &\equiv -\alpha_\delta^*\frac{1}{\gamma r}\ln\overline{\lambda}.
\end{aligned}
$$

The free boundary $\overline{\lambda}$ has lower and upper bounds of the following form:

$$
\lambda_\delta^0 \le \overline{\lambda} \le \lambda_\delta^1, \tag{22}
$$

where λ_δ^0 and λ_δ^1 satisfy

$$
\overline{\phi}_\delta(\lambda_\delta^0) = L_\delta, \quad \textit{and} \quad \underline{\phi}_\delta(\lambda_\delta^1) = L_\delta. \tag{23}
$$

Proof. Obviously,

$$
\underline{\phi}_\delta(\overline{\lambda}) \le \psi_\delta(\overline{\lambda}) \le \overline{\phi}_\delta(\overline{\lambda}).
$$

Therefore, λ_δ^0 (λ_δ^1) becomes a lower (an upper) bound for $\overline{\lambda}$, since both $\overline{\phi}_\delta(\overline{\lambda})$ and $\underline{\phi}_\delta(\overline{\lambda})$ are monotonically-increasing and continuous functions with

$$\overline{\phi}_\delta(0) = \underline{\phi}_\delta(0) = -\infty, \quad \text{and}$$
$$\overline{\phi}_\delta(+\infty) = \underline{\phi}_\delta(+\infty) = +\infty. \quad \textbf{Q.E.D.}$$

Theorem 7.3 *(Existence) Suppose that the inequality of (21) in Theorem 7.2 holds. Then, there exists a unique solution* $\overline{\lambda}$ *of (18) satisfying* $0 < \overline{\lambda} \leq 1$*. The corresponding solution* $G(\lambda)$ *of (16) is uniquely determined.*

Proof. The inequality of (22), the equality of (23), and the continuity of $\psi_\delta(\overline{\lambda})$ yield the fact that there exists at least one solution of

$$\psi_\delta(\overline{\lambda}) = L_\delta. \tag{24}$$

Using the assumption of

$$\frac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)\lambda_\delta^0} < 1, \tag{25}$$

we know that any solution of (24) satisfies the assumption in Theorem 7.1. From the definition of λ_δ^0 (see equation (23)), the inequality (25) can be rewritten as

$$\overline{\phi}_\delta\Big(\frac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\Big) < L_\delta.$$

The definition of $\overline{\phi}_\delta(\cdot)$ in Proposition 7.1 gives us the following relationship:

$$\frac{1}{\gamma r}(1 + \alpha_\delta^* \ln \lambda_\delta^0) + \alpha_\delta^*\Big\{\frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta + \nu + \delta - r)\Big)\Big\} \geq 0. \tag{26}$$

Notice that, if the inequality of (21) holds and $0 < \overline{\lambda} \leq 1$, the inequality of (26) is true. Hence, we can say that there exists a unique solution $\overline{\lambda}$ of (18) by Theorem 7.2 if the inequality of (21) holds and $0 < \overline{\lambda} \leq 1$ is satisfied. Moreover, the function $G(\lambda)$ which can be obtained by utilizing the solution $\overline{\lambda}$ of (18) becomes the unique solution of (16) by Theorem 7.1.

Now, it remains to verify whether the unique solution $\overline{\lambda}$ satisfies $0 < \overline{\lambda} \leq 1$ or not. Notice that the inequality is a sufficient condition of $\overline{\lambda} \leq 1$, which can be restated as

$$\underline{\phi}_\delta(1) \leq L_\delta.$$

From the definition of $\underline{\phi}_\delta(\cdot)$ in Proposition 7.1, we can obtain

$$-\frac{2\delta A}{\theta^2\gamma} \leq \frac{1}{\gamma r} + \alpha_\delta^*\Big\{\frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta + \nu + \delta - r)\Big)\Big\}.$$

The above inequality is true all the time if the inequality of (21) holds. **Q.E.D.**

7.4 Convergence of the Iterative Numerical Algorithm

In this section, we show that the approximation function G obtained from the iterative numerical algorithm given in Appendix of How Annuity Demand is Affected by Insurer Default Risk converges to the implicit equation (16) by using the Banach fixed-point theorem.

Consider the domain of $\lambda(x)$ as a set $D = [0, \overline{\lambda}]$. Let $\mathbf{R}$ be the set of real numbers. The set $\mathbf{B}(D, \mathbf{R})$ of all bounded functions $f : D \to \mathbf{R}$ is a complete metric space with the supremum norm

$$d(f, g) \equiv \sup\{|f(x) - g(x)| : x \in D\},$$

because the set $\mathbf{R}$ is complete. Denote $\mathbf{C}(D, \mathbf{R})$ by the set of all continuous bounded functions $f : D \to \mathbf{R}$. Then the set $\mathbf{C}(D, \mathbf{R})$ is a closed subspace of $\mathbf{B}(D, \mathbf{R})$ and subsequently, $\mathbf{C}(D, \mathbf{R})$ is also a complete metric space. Therefore, the continuous and bounded function $G(\lambda)$ on D is in $\mathbf{C}(D, \mathbf{R})$.

We define a map M

$$\begin{aligned} M\big(G(\lambda)\big) \equiv & -\frac{1}{\gamma r}\ln\lambda - \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta + \nu + \delta - r)\Big) + B(\overline{\lambda})\lambda^{-\alpha_\delta^*} \\ & - \frac{2\delta A}{\theta^2(\alpha_\delta - \alpha_\delta^*)\gamma}\Big[(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta - 2} e^{-\gamma\left(r(G(\mu) - \frac{\epsilon}{r}) + k\epsilon\right)}\,d\mu \\ & + (\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^* - 2} e^{-\gamma\left(r(G(\mu) - \frac{\epsilon}{r}) + k\epsilon\right)}\,d\mu\Big] \end{aligned}$$

for any $G(\lambda) \in \mathbf{C}(D, \mathbf{R})$. Then the map M is continuous, as a result, $T\big(G(\lambda)\big)$ is in $\mathbf{C}(D, \mathbf{R})$, because

$$\big|M\big(G(\lambda)\big)\big| \leq \frac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}}.$$

When we take the assumption of

$$\frac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}} < 1,$$

the map $M : \mathbf{C}(D, \mathbf{R}) \to \mathbf{C}(D, \mathbf{R})$ is a contraction mapping. This is induced by the fact that for any $G_1(\lambda)$, $G_2(\lambda) \in \mathbf{C}(D, \mathbf{R})$,

$$\sup_\lambda \big|M\big(G_1(\lambda)\big) - M\big(G_2(\lambda)\big)\big| = \frac{2\delta A r}{\theta^2(\alpha_\delta - \alpha_\delta^*)\overline{\lambda}} \sup_\lambda |G_1(\lambda) - G_2(\lambda)|.$$

Denote $G^i(\lambda)$ by a function and $B^i(\overline{\lambda})$, $\overline{\lambda}^i$ by the two constants obtained from the i-th iteration. Then we conclude that $G^i(\lambda)$ converges uniformly to $G(\lambda)$ given by (16) on D if we apply the Banach fixed-point theorem. Trivially, $B^i(\overline{\lambda}) \to B(\overline{\lambda})$ and $\overline{\lambda}^i \to \overline{\lambda}$ as $i \to \infty$.

7.5 The Details of Solutions in Robustness Section

In robustness section, we relax the assumption by allowing the bequest motive of a retiree, a correlation between stock price and annuity default event, and time-varying mortality rates. To consider the bequest motive, the retiree's problem follows

$$V(x) = \max_{(c,\pi)} E\Big[\int_0^\infty e^{-(\beta+\nu+\delta)t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t} - \delta\frac{A}{\gamma}e^{-\gamma(rX_t+k\epsilon)} - \tilde{k}\nu\frac{A}{\gamma}e^{-\gamma rX_t}\Big)dt\Big],$$

where $\tilde{k} > 0$ measures the intensity of preference for leaving the bequest to heirs of the retiree. Then the HJB equation is given by

$$-(\beta+\nu+\delta)V(x) + (rx+\epsilon)V'(x) - \frac{1}{2}\theta^2\frac{V'(x)^2}{V''(x)} + \frac{1}{\gamma}V'(x)\{\ln V'(x) - 1\}$$
$$- \frac{A\delta}{\gamma}e^{-\gamma(rx+k\epsilon)} - \frac{A\tilde{k}\nu}{\gamma}e^{-\gamma rx} = 0.$$

If we differentiate the HJB equation with respect to x, then we get

$$\begin{aligned}-(\beta+\nu+\delta)V'(x) + rV'(x) + (rx+\epsilon)V''(x) - \frac{1}{2}\theta^2\frac{2V'(x)V''(x)^2 - V'(x)^2V'''(x)}{V''(x)^2}\\ + \frac{1}{\gamma}V''(x)\ln V'(x) + Ar\delta e^{-\gamma(rx+k\epsilon)} + Ar\tilde{k}\nu e^{-\gamma rx} = 0.\end{aligned} \tag{27}$$

Define

$$\lambda(x) \equiv V'(x)$$

and introduce the dual function G

$$G\big(\lambda(x)\big) \equiv x + \frac{\epsilon}{r}.$$

Then the HJB equation (27) is rewritten as

$$\begin{aligned}-\frac{1}{2}\theta^2\lambda^2G''(\lambda) - \lambda G'(\lambda)\{\theta^2+\beta+\nu+\delta-r\} + rG(\lambda)\\ + Ar\delta e^{-\gamma\left(r(G(\lambda)-\frac{\epsilon}{r})+k\epsilon\right)}G'(\lambda) + Ar\tilde{k}\nu e^{-\gamma\left(r(G(\lambda)-\frac{\epsilon}{r})\right)} = -\frac{1}{\gamma}\ln\lambda.\end{aligned} \tag{28}$$

We derive the general solution to (28) as the following:

$$
\begin{aligned}
G(\lambda) = &-\frac{1}{\gamma r}\ln\lambda - \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big) + B(\overline{\lambda})\lambda^{-\alpha_\delta^*} \\
&- \frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}\Big[(\alpha_\delta-1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu \\
&+(\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}}\mu^{\alpha_\delta^*-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu\Big] \\
&- \frac{2\tilde{k}\nu A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}\Big[(\alpha_\delta-1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})\right)}d\mu \\
&+(\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}}\mu^{\alpha_\delta^*-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})\right)}d\mu\Big],
\end{aligned}
$$

where $\overline{\lambda}$ and $B(\overline{\lambda})$ are two constants to be determined by (13). By using the borrowing constraints (13), we obtain

$$
\begin{aligned}
\frac{\epsilon}{r} = &-\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big) + B(\overline{\lambda})\overline{\lambda}^{\,-\alpha_\delta^*} \\
&- \frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}(\alpha_\delta-1)\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu \\
&- \frac{2\tilde{k}\nu A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma}(\alpha_\delta-1)\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})\right)}d\mu,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\gamma r}+\frac{\alpha_\delta^*\epsilon}{r}+\frac{\theta^2\alpha_\delta^*}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\nu+\delta-r)\Big) = &-\alpha_\delta^*\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{2\delta A}{\theta^2\gamma}\frac{1}{\overline{\lambda}} \\
&+\frac{2\delta A(\alpha_\delta-1)}{\theta^2\gamma}\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}}\mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu \\
&+\frac{2\tilde{k}\nu A(\alpha_\delta-1)}{\theta^2\gamma}\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}}\mu^{\alpha_\delta-2}e^{-\gamma\left(r(G(\mu)-\frac{\epsilon}{r})+k\epsilon\right)}d\mu.
\end{aligned}
$$

By using the iterative numerical algorithm given in Appendix of the paper of How Annuity Demand is Affected by Insurer Default Risk, the two constants of $\overline{\lambda}$ and $B(\overline{\lambda})$ are determined.

The retiree's problem for this case is given by

$$
V(x) = \max_{(c,\pi)} E^Q\Big[\int_0^\infty e^{-(\beta+\nu+\delta)}\Big(-\frac{1}{\gamma}e^{-\gamma c_t} - \delta\frac{A}{\gamma}e^{-\gamma(rX_t+k\epsilon)}\Big)dt\Big],
$$

where Q is the probability defined by

$$
dQ = e^{bW_t^*+\tilde{b}\widetilde{W}_t-\frac{1}{2}(b^2+\tilde{b}^2)t}dP.
$$

Let $a=-\delta$ and $\tilde{a}=-\nu$ and define

$$
W_t^Q = W_t - (\rho b+\tilde{b})t.
$$

Then, W_t^Q is also a Brownian motion under Q and the wealth process follows

$$dX_t = \begin{cases} \big(rX_t - c_t + \epsilon\big)dt + \pi_t\sigma\big(dW_t + (\theta + \rho b + \tilde{b})dt\big), & 0 \le t < \tau \wedge \tau_M, \\ \big(rX_t - c_t + k\epsilon\big)dt + \pi_t\sigma\big(dW_t + (\theta + \rho b + \tilde{b})dt\big), & \tau \wedge \tau_M \le t \le \tau_M. \end{cases}$$

We can obtain the following HJB equation for this case:

$$-(\beta + \nu + \delta)V(x) + (rx + \epsilon)V'(x) - \frac{1}{2}(\theta + \rho b + \tilde{b})^2 \frac{V'(x)^2}{V''(x)} + \frac{1}{\gamma}V'(x)\{\ln V'(x) - 1\} - \frac{A\delta}{\gamma}e^{-\gamma(rx + k\epsilon)} = 0.$$

Notice that the HJB equation has the same form as (10) except that θ is replaced by $\theta + \rho b + \tilde{b}$. Hence, we can obtain a solution to the problem by utilizing the same arguments described in Section 7.2.

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	$k=0$				$k=0.1$				$k=0.25$			
$x \setminus \delta$	Aaa	Aa	A	B	Aaa	Aa	A	B	Aaa	Aa	A	B
1	1.1371	1.1300	1.1260	1.0469	1.1326	1.1311	1.1288	1.0930	1.1327	1.1323	1.1322	1.1664
10	2.0755	2.0596	2.0366	1.7398	2.0748	2.0636	2.0461	1.8023	2.0752	2.0681	2.0576	1.8894
20	2.7194	2.6871	2.6410	2.1895	2.7195	2.6945	2.6575	2.2612	2.7203	2.7030	2.6778	2.3641
30	3.2571	3.2064	3.1379	2.5806	3.2578	3.2173	3.1606	2.6578	3.2590	3.2301	3.1892	2.7841
40	3.7427	3.6732	3.5845	2.9552	3.7439	3.6875	3.6125	3.0359	3.7456	3.7044	3.6484	3.1890
50	4.1971	4.1093	4.0034	3.3295	4.1988	4.1268	4.0357	3.4116	4.2009	4.1477	4.0777	3.5735

(A) consumption

	$k=0$				$k=0.1$				$k=0.25$			
$x \setminus \delta$	Aaa	Aa	A	B	Aaa	Aa	A	B	Aaa	Aa	A	B
1	4.9360	4.9024	4.9514	5.9681	4.8716	4.8936	4.9301	5.9793	4.8701	4.8843	4.9078	6.2262
10	13.2873	13.5563	13.9905	19.1176	13.2675	13.4948	13.8481	18.6576	13.2612	13.4244	13.6769	18.0992
20	17.0904	17.6332	18.3955	24.0844	17.0719	17.5228	18.1680	23.6951	17.0598	17.3932	17.8819	22.6893
30	19.4330	20.1824	21.1097	25.9751	19.4125	20.0400	20.8520	25.6721	19.3955	19.8677	20.5128	23.9466
40	21.0595	21.9355	22.8843	26.4341	21.0367	21.7729	22.6365	26.2535	21.0165	21.5840	22.2962	24.7958
50	22.1026	23.0891	24.0704	26.0817	22.0784	22.9668	23.8046	26.0728	22.0596	22.8175	23.5572	26.6611

(B) risky investment

Table 1: **Optimal consumption c_t and risky investment π_t as a function of initial wealth x for different levels of default risk.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, and $\nu = 0.05$.

$x \setminus \delta$	$\mu = 0.1023$				$\mu = 0.1223$			
	Aaa	Aa	A	B	Aaa	Aa	A	B
1	1.1397	1.1296	1.1204	0.9541	1.1396	1.1341	1.1337	1.1250
10	2.0171	1.9911	1.9534	1.5450	2.1405	2.1317	2.1202	1.9384
20	2.6256	2.5769	2.5103	1.9599	2.8213	2.8013	2.7732	2.4339
30	3.1374	3.0657	2.9745	2.3418	3.3860	3.3523	3.3059	2.8483
40	3.6026	3.5089	3.3979	2.7239	3.8933	3.8446	3.7800	3.2357
50	4.0401	4.9264	3.7998	3.1134	4.3663	4.3021	4.2205	3.6149

(A) expected rate of return μ

$x \setminus \delta$	$\sigma = 0.1854$				$\sigma = 0.2054$			
	Aaa	Aa	A	B	Aaa	Aa	A	B
1	1.1376	1.1313	1.1288	1.0809	1.1375	1.1294	1.1237	1.0141
10	2.1011	2.0885	2.0705	1.8200	2.0533	2.0340	2.0061	1.6660
20	2.7598	2.7331	2.6946	2.2867	2.6841	2.6461	2.5928	2.1022
30	3.3084	3.2652	3.2057	2.6866	3.2122	3.1541	3.0773	2.4900
40	3.8026	3.7421	3.6629	3.0660	3.6901	3.6119	3.5149	2.8764
50	4.2642	4.1863	4.0897	3.4419	4.1382	4.0409	3.9270	3.2618

(B) volatility σ

Table 2: **Optimal consumption c_t as a function of initial wealth x for changes in expected return and volatility of the risky asset.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

$x \setminus \delta$	$\mu = 0.1023$				$\mu = 0.1223$			
	Aaa	Aa	A	B	Aaa	Aa	A	B
1	4.6083	4.6008	4.6921	6.3022	5.2137	5.1598	5.1796	5.6581
10	12.2427	12.6353	13.2321	18.5436	14.1900	14.3569	14.6376	19.1435
20	15.6176	16.3101	17.1955	21.9488	18.3821	18.7741	19.3556	25.3606
30	17.6414	18.5087	19.4431	22.5046	21.0210	21.6157	22.4205	28.1503
40	19.0076	19.9411	20.7929	22.1290	22.8832	23.6342	24.5467	29.2658
50	19.9950	20.9138	21.6320	21.7609	24.3920	24.8106	26.0184	29.4576

(A) expected rate of return μ

$x \setminus \delta$	$\sigma = 0.1854$				$\sigma = 0.2054$			
	Aaa	Aa	A	B	Aaa	Aa	A	B
1	5.3267	5.2817	5.3195	6.1493	4.5876	4.5641	4.6239	5.8030
10	14.4064	14.6422	15.0325	20.2444	12.2935	12.5910	13.0607	18.0901
20	18.5858	19.0906	19.8228	26.0158	15.7667	16.3373	17.1138	22.1307
30	21.1847	21.9097	22.8493	28.3619	17.8866	18.6467	19.5456	23.2016
40	23.0039	23.8818	24.8947	29.1121	19.3465	20.2071	21.0914	23.2062
50	24.3477	25.2899	26.3016	28.9867	20.3593	21.2445	22.0813	23.0758

(B) volatility σ

Table 3: **Optimal risky investment π_t as a function of initial wealth x for changes in expected return and volatility of the risky asset.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

$x \setminus \delta$	$\mu = 0.1023$				$\mu = 0.1223$			
	Aaa	Aa	A	B	Aaa	Aa	A	B
1	10.5969	10.3207	9.9773	5.8013	8.9494	8.8274	8.6881	5.8009
10	17.1402	16.0808	14.6881	6.4035	15.2702	14.8500	14.2266	6.2944
20	20.0226	17.9467	15.5583	6.3582	18.2691	17.5641	16.0527	6.2347
30	21.6847	18.5960	15.4657	5.9268	20.1205	18.5641	16.5974	5.9836
40	22.7588	18.7421	15.1350	4.9828	21.4080	19.1564	16.5672	5.3107
50	23.3894	18.6236	14.8167	3.9411	22.7702	19.6033	15.8081	4.4455

(A) expected rate of return μ

$x \setminus \delta$	$\sigma = 0.1854$				$\sigma = 0.2054$			
	Aaa	Aa	A	B	Aaa	Aa	A	B
1	9.4016	9.2464	9.0637	6.1412	10.0353	9.8229	9.5622	5.9526
10	15.8060	15.2390	14.4267	6.9387	16.5279	15.7182	14.6081	6.5752
20	18.7859	17.5727	15.9802	6.5698	19.4642	17.8115	15.7848	6.3086
30	20.5924	18.6349	16.3074	6.4019	21.1988	18.6476	15.8603	5.8816
40	21.8141	19.0775	16.1450	6.2084	22.3479	18.9013	15.5731	5.2424
50	22.8009	19.2488	15.7963	5.7634	25.4318	19.1624	15.2520	4.7042

(B) volatility σ

Table 4: **Implicit value of life annuity as a function of initial wealth x for changes in expected return and volatility of the risky asset.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

	$\mu = 0.1023$				$\mu = 0.1223$			
$x \setminus \delta$	Aaa	Aa	A	B	Aaa	Aa	A	B
1	0.0165	0.2396	0.5522	2.8077	0.0001	0.0576	0.1409	2.0762
10	0.0060	0.0700	0.1583	0.8803	0.0018	0.0249	0.0598	0.4711
20	0.0051	0.0560	0.1225	0.6068	0.0020	0.0243	0.0564	0.3720
30	0.0046	0.0495	0.1053	0.4932	0.0021	0.0244	0.0554	0.3216
40	0.0043	0.0450	0.0935	0.4346	0.0022	0.0245	0.0545	0.2914
50	0.0041	0.0413	0.0843	0.3991	0.0022	0.0245	0.0532	0.2731

(A) expected rate of return μ

	$\sigma = 0.1854$				$\sigma = 0.2054$			
$x \setminus \delta$	Aaa	Aa	A	B	Aaa	Aa	A	B
1	0.0030	0.0956	0.2280	0.9623	0.0094	0.1636	0.3836	2.1633
10	0.0028	0.0355	0.0833	0.5868	0.0044	0.0527	0.1211	0.7435
20	0.0028	0.0323	0.0735	0.4392	0.0039	0.0445	0.0990	0.5302
30	0.0028	0.0311	0.0691	0.3693	0.0037	0.0408	0.0884	0.4381
40	0.0028	0.0298	0.0656	0.3298	0.0036	0.0382	0.0809	0.3886
50	0.0028	0.0296	0.0625	0.3069	0.0035	0.0360	0.0746	0.3576

(B) volatility σ

Table 5: **Certainty equivalent wealth gain as a function of initial wealth x for changes in expected return and volatility of the risky asset.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

$x \setminus \delta$	$\tilde{k} = 0.05$				$\tilde{k} = 0.10$			
	Aaa	Aa	A	B	Aaa	Aa	A	B
1	9.5003	9.3663	9.1596	6.0033	9.4633	9.3324	9.1338	5.9948
10	14.8535	14.3205	13.5556	6.7152	13.8855	13.4511	12.8149	6.6644
20	16.3868	15.4519	14.1835	6.0891	14.6235	13.9374	12.9623	6.0651
30	16.7176	15.4571	13.8636	5.3609	14.3854	13.5258	12.3259	5.2911
40	16.2655	15.1029	13.2894	5.0268	13.8878	12.9295	11.6536	4.8108
50	16.4869	14.8562	12.4398	5.3161	13.4955	12.5640	11.3236	4.9620

Table 6: **Implicit value of life annuity as a function of initial wealth x for changes in $\tilde{k}$.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

	A-rated annuity provider			B-rated annuity provider		
	$b = 0$	$b = 0.1$		$b = 0$	$b = 0.1$	
$x \setminus \rho$		0.1	−0.1		0.1	−0.1
1	9.32452	9.9536	9.3245	6.0474	6.1378	6.0473
10	14.5331	14.5217	14.5331	6.6683	6.9284	6.6686
20	15.8903	16.0435	15.8911	6.4155	6.5633	6.4170
30	16.0815	16.5805	16.0822	6.3911	6.4006	6.3928
40	15.8504	16.8132	15.8528	6.2640	6.2085	6.2639
50	15.4103	16.9610	15.5094	5.7710	5.7728	5.7714

Table 7: **Implicit value of life annuity as a function of initial wealth x for changes in b and ρ.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, $k = 0$, $a = -\delta$, $\tilde{a} = -\nu$, and $\tilde{b} = 0$.

	$\tilde{b} = 0$				$\tilde{b} = 0.01$				$\tilde{b} = 0.1$			
$x \setminus \delta$	Aaa	Aa	A	B	Aaa	Aa	A	B	Aaa	Aa	A	B
1	9.7280	9.5453	9.3245	6.0474	9.5689	9.4001	10.0901	6.0931	8.2844	8.1957	8.1076	6.3689
10	16.1819	15.4955	14.5331	6.6683	15.9998	15.3731	14.5757	6.7959	14.4507	14.1964	13.8087	7.7869
20	19.1418	17.7080	15.8903	6.4155	18.9701	17.6457	16.0195	6.4859	17.4572	16.8727	16.0107	7.4839
30	20.9127	18.6520	16.0815	6.3911	20.7589	18.6448	16.5184	6.3907	19.3565	18.3307	16.9158	6.8921
40	22.1047	19.0110	15.8504	6.2640	21.9656	19.0496	16.7475	6.2285	20.7039	19.1491	17.1548	6.6609
50	23.0345	18.9418	15.4103	5.7710	22.9878	19.0529	16.9287	5.7708	22.3681	19.6576	17.0910	6.8202

Table 8: **Implicit value of life annuity as a function of initial wealth x for changes in $\tilde{b}$.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, $k = 0$, $a = -\delta$, $\tilde{a} = -\nu$, $b = 0$, and $\rho = 0$.

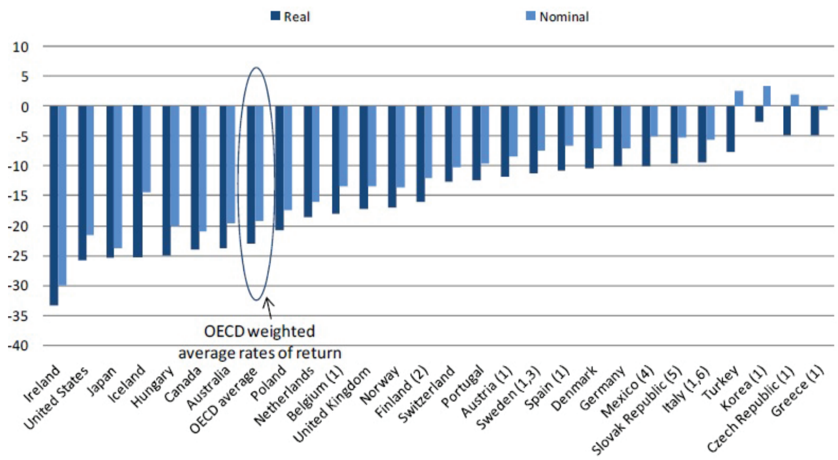

Figure 1: **Pension fund returns in selected OECD countries (January-October 2008)**. Source: Impavido and Tower (2009).

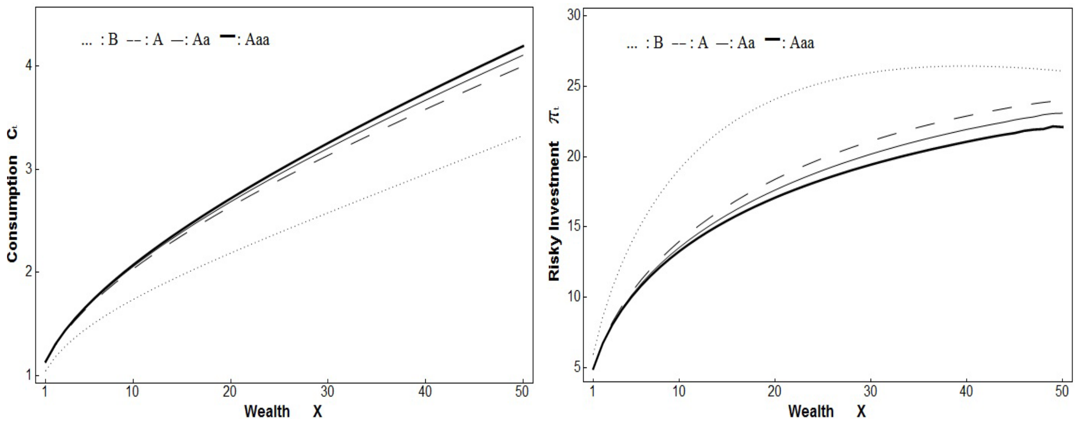

Figure 2: **Optimal consumption and risky investment as a function of wealth** x. Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

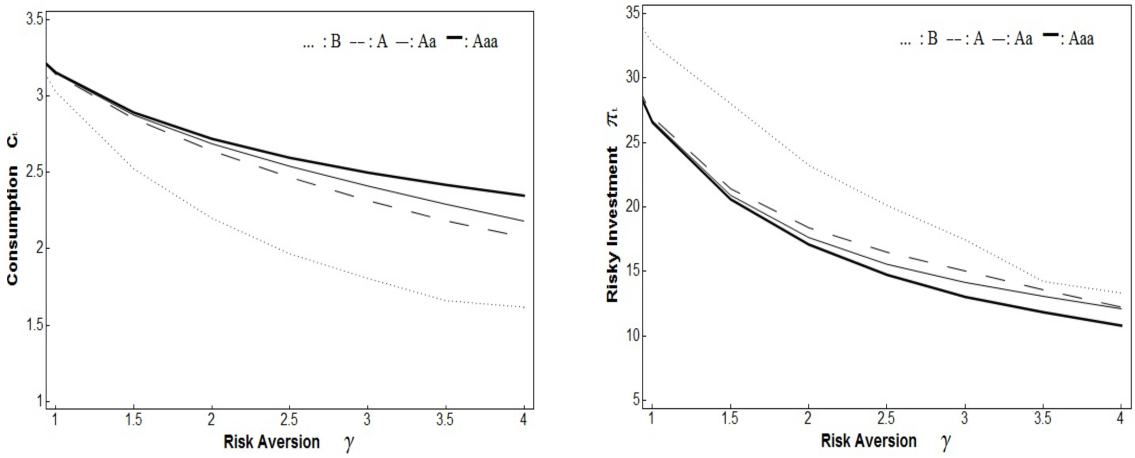

Figure 3: **Optimal consumption and risky investment as a function of risk aversion**. Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1223$, $\sigma = 0.1954$, $x = 20$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

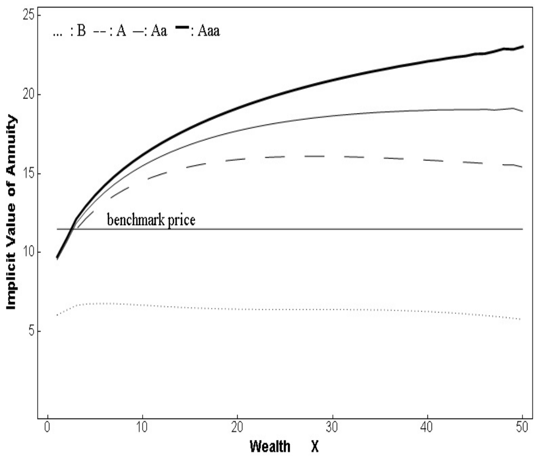

Figure 4: **Implicit value of annuity as a function of wealth**. Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

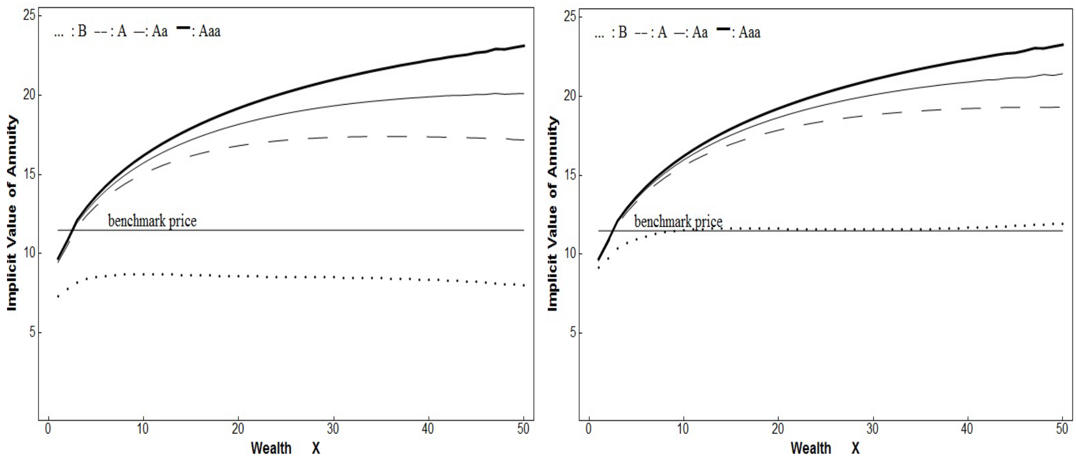

Figure 5: **Implicit value of annuity as a function of wealth (Left figure:** $k = 0.10$**, Right figure:** $k = 0.25$. Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, and $\nu = 0.05$.

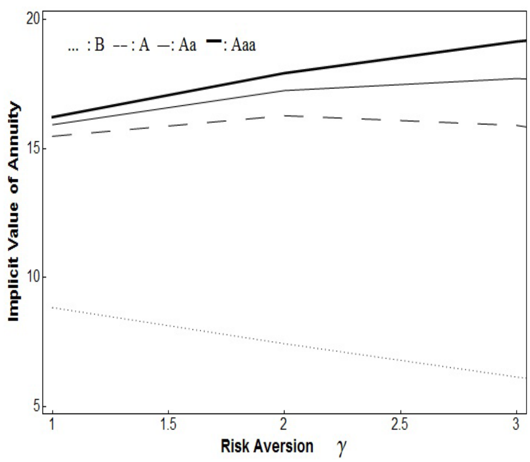

Figure 6: **Implicit value of annuity as a function of risk aversion**. Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $x = 20$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0.1$.

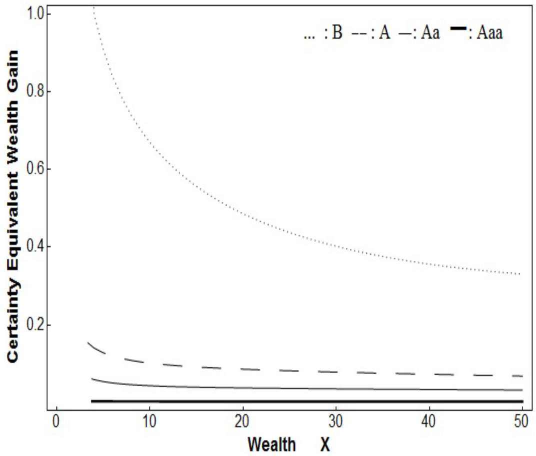

Figure 7: **Certainty equivalent wealth gain to wealth ratio $\Delta(x)/x$ as a function of wealth x.** Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

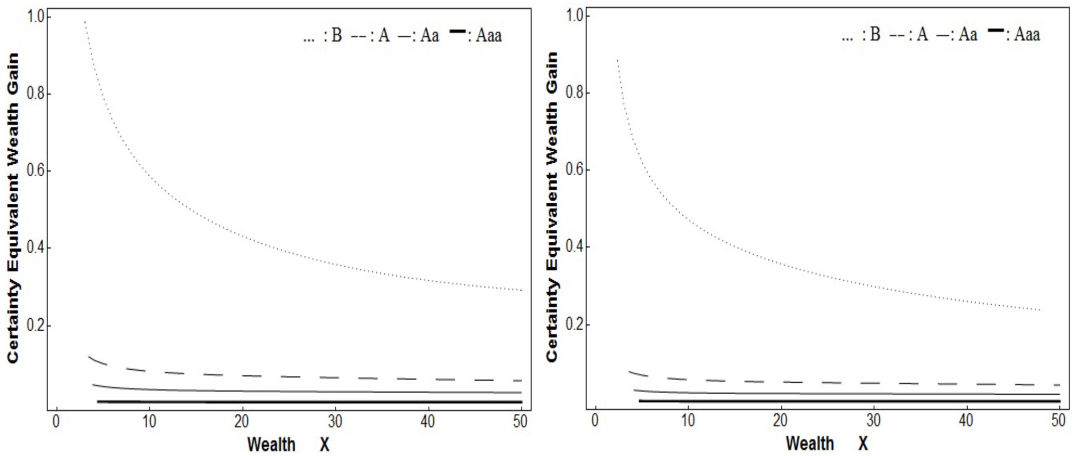

Figure 8: **Certainty equivalent wealth gain to wealth ratio $\Delta(x)/x$ as a function of wealth x (Left figure: $k = 0.10$, Right figure: $k = 0.25$.** Parameter values are set as follows: $\delta = 0.03$, $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\epsilon = 1$, and $\nu = 0.05$.

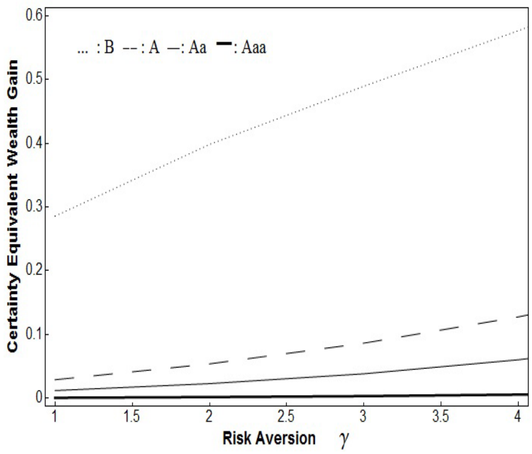

Figure 9: **Certainty equivalent wealth gain to wealth ratio $\Delta(x)/x$ as a function of risk aversion**. Parameter values are set as follows: $r = 0.0371$, $\beta = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $x = 20$, $\epsilon = 1$, $\nu = 0.05$, and $k = 0$.

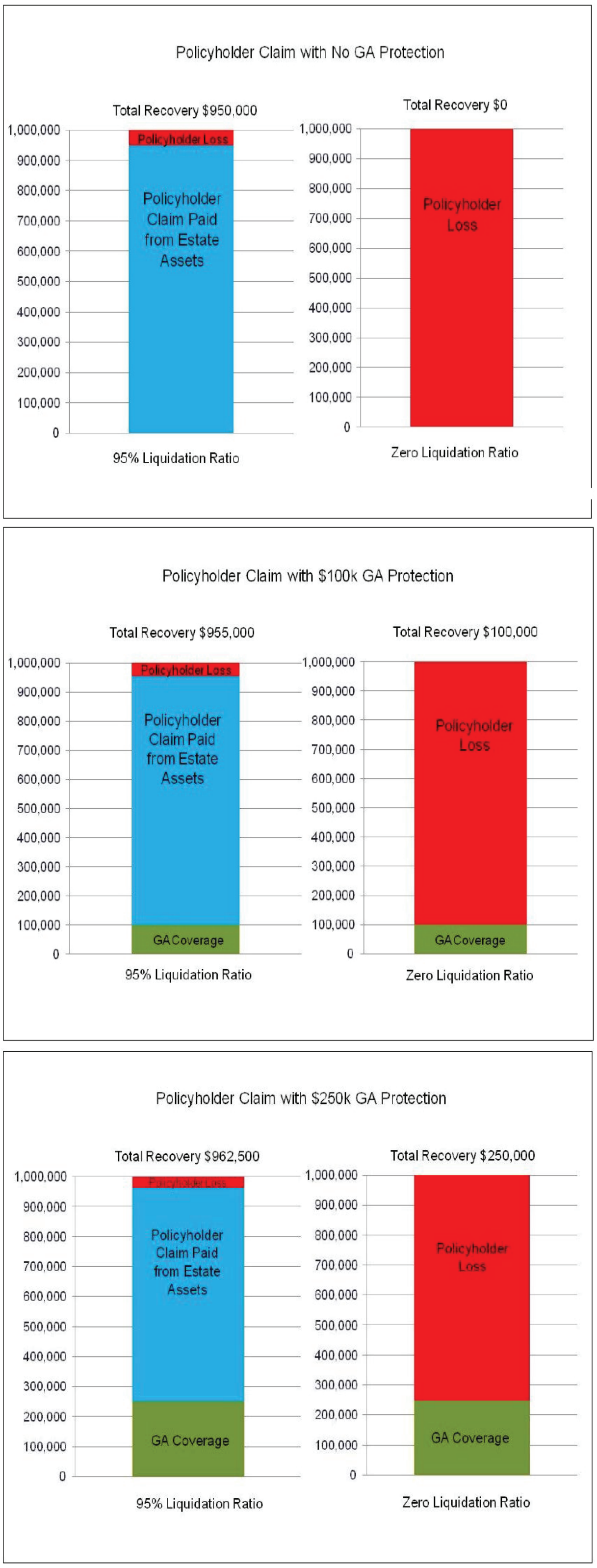

Figure 10: **The recovery amount for the insurance insolvency.** Source: NOLHGA (2011).