Coexisting Exchange Platforms: Limit Order Books and Automated Market Makers

Jun Aoyagi[*]　　　Yuki Ito[†]

First Draft: February 2021

This Version: March 2021

Abstract

A growing number of blockchain-based decentralized exchanges (DEX) have adopted Constant Function Market Makers (CFMMs)—a single-function algorithm to determine the execution price for a trade. We build a model of coexisting exchanges where a centralized exchange (CEX) with the traditional order-book mechanism operates in parallel with a DEX with the CFMM. Traders are either informed or uninformed and endogenously choose their trading venue. We first investigate how the arrival of the CFMM affects an adverse selection cost for market makers and liquidity on both exchanges. Our result indicates that liquidity on the DEX has a positive spillover effect on CEX liquidity. Secondly, we derive a profit function for liquidity providers using the CFMM when there is an asymmetric information problem. As in the traditional market microstructure theory, informed trading imposes an adverse selection cost, while uninformed noise trading adds value to liquidity pools. We analyze the market makers' equilibrium behavior and endogenize the amount of liquidity supplied via the CFMM.

Key words: Automated Market Makers, Constant Function Market Makers, DeFi, decentralized exchanges, Uniswap, limit order book, market liquidity, asymmetric information

JEL code:

[*]Department of Finance, The Hong Kong University of Science and Technology. Email: junaoyagi@ust.hk.

[†]Department of Economics, University of California at Berkeley. Email: yuki_ito@berkeley.edu.

We appreciate the constructive comments from Tarun Chitra, Kei Kawai, and Cristian Ugarte, as well as participants at PHDBA 297T and Finance Lunch at UC Berkeley, and the Digital Finance Workshop at the University of Tokyo.

1 Introduction

A limit order book has been a core trading mechanism in the modern electronic financial market. Traders called market makers provide trading opportunities by placing *limit orders*—a quote at which they are willing to buy or sell a certain amount of an asset. Limit orders are stored on a book (i.e., a limit order book, an LOB) and publicly displayed. If other traders find a good deal on the book, they try to consume the trading opportunities by placing marketable limit orders or *market orders*.[1] An incoming market order is matched with standing limit orders on the book, and a trade is executed at the proposed price.

The recent hype in cryptocurrency and blockchain, however, changes the landscape of market structure. In particular, many exchange platforms have been built on smart contracts on the Ethereum blockchain, and transactions are executed in a decentralized manner. Those platforms are called decentralized exchanges (DEXs).[2] As Figure 1 illustrates, trading volume on DEXs has been exponentially growing—it hits the record high in February 2021 (about $74B monthly volume), obtaining more than 10% of trading share to the traditional centralized exchanges (CEXs) for digital tokens. DEXs are based on a trustless record-keeping system run by the network of innumerable computer nodes on the blockchain and robust to cyber attacks and a single point of failure.

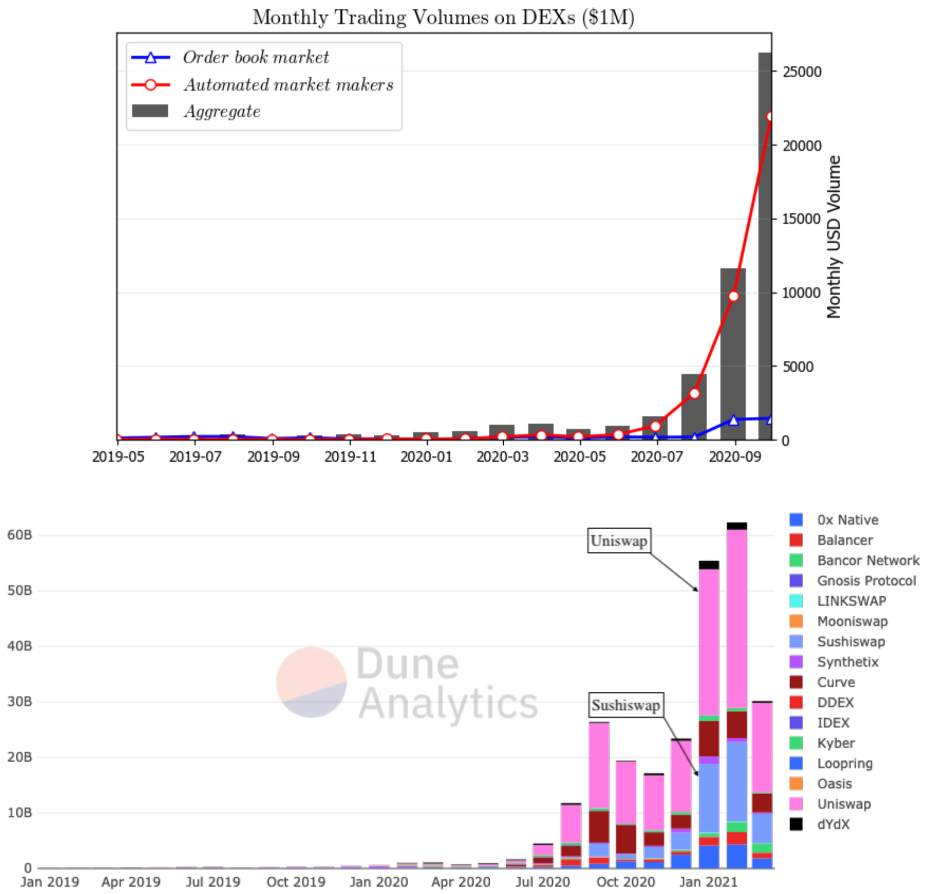
From the market-microstructure perspective, DEXs have proposed and implemented a novel pricing and matching algorithm called *automated market makers* (AMMs),[3] and they play a substantial role in the prosperity of DEXs, as shown by the lower panel of Figure 1. An automated market maker reserves traded assets as *liquidity pools* and determines asset prices (or exchange rates) with a single-function algorithm by taking the state of the pools as an input. A trader takes liquidity by trading against the pools, and it does not require the physical presence of active market makers or dealers

[1] A huge variety of sophisticated order types is used in the modern financial market, as analyzed by Li, Ye and Zheng (2021), but it suffices to consider market and limit orders in our model.

[2] Many decentralized exchanges (DEXs) attain a sizable trading share for cryptocurrency, such as Uniswap, Curve, IDEX, 0x, Waves, EtherDelta, dYdX, Balancer, Sushiswap, and so on, where all transactions are settled P2P through users' wallets and smart contracts. In the traditional centralized exchanges (CEXs), such as Coinbase, Bittrex, and Binance, a centralized authority manages trader information and funds. Although CEXs have achieved large liquidity and trading volume, they are vulnerable to cyber-attacks and single point of failures. In contrast, information on DEXs is recorded on the blockchain with KYC (Know Your Customer) data and stay immutable, providing users with an efficient and robust transaction method.

[3] The idea of AMMs has been discussed in the context of information markets, such as Hanson (2003).

Figure 1: Monthly trading volume on DEXs



Source: Dune Analytics (duneanalytics.com)

for pricing and order execution. As a result, an AMM requires much smaller memory than the traditional order-book algorithm and allows a substantial part of trades to be on the blockchain. There are several types of AMMs but, as shown by the lower panel of Figure 1, Constant Product Market Makers (CPMMs) proposed by Uniswap (and adopted by Sushiswap) have been a dominant market structure—more than 70% of transactions on the DEXs are handled by CPMMs. In light of this, our paper focuses on CPMMs and a more general class of AMMs, called Constant Function Market Makers (CFMMs).

On an automated market, liquidity providers lock traded assets into the exchange, and AMMs aggregate them to create liquidity pools. Suppose that the liquidity pools reserve x and y unit of token X and token Y prior to a trade. If a trader buys δ unit of token Y by paying $p\delta$ of token X, she subtracts token Y from the pool and adds price-adjusted token X to the pool, triggering a change in the liquidity pools from (x, y) to $(x', y') = (x + p\delta, y - \delta)$. Then the CPMM algorithm requires the (squared) geometric mean of the liquidity pools to be constant, $k = xy = x'y'$, with some pre-determined k. This single equation derives the price of token Y in terms of X (or the exchange rate) for this trade as $p = \frac{x}{y-\delta}$. CFMMs are a generalized form of CPMMs that determine the price by imposing $f(x', y') = f(x, y)$ with a certain function f that satisfies some regularity conditions (the CPMM is a special case with $f(x, y) = xy$). When a liquidity provider exits the market, she withdraws and liquidates her contribution to the pools. Thus, the accumulated gain or loss in the pools' value caused by trade δ (i.e., $x' - x$ and $y' - y$) is distributed to liquidity providers on a pro rata basis. In general, liquidity on an automated market is measured by the amount of assets locked in the platform, i.e., x and y.

This paper studies how the introduction of CFMMs affects the liquidity of the entire market when traders face an asymmetric information problem. Importantly, our model features the coexisting exchanges with two different market-making algorithms: a DEX with the CFMM and a CEX with a limit order book. There are informed for-profit traders, uninformed liquidity traders, and market makers, and they are endogenously differentiated between two exchanges. As the traditional theory of market microstructure suggests, the bid-ask spread on the CEX reflects the cost of asymmetric information for uninformed market makers. We first analyze the consequence of an exogenous variation in the DEX liquidity for traders' behavior—in particular, their venue choice—and its impact on the CEX liquidity. We then endogenize liquidity provision by market makers on the DEX and describe how market liquidity on the DEX and the CEX jointly reacts to a more severe informational friction.

Our model shows that ample liquidity on the DEX complements that on the CEX. A larger liquidity pool on the DEX mitigates the price impact of a liquidity-taking order. Importantly, informed traders can enjoy this effect more than liquidity traders do. On the one hand, an informed trader is informed of the (future) asset value and anticipates the trading direction of other informed traders. Thus, they tend to cluster on the same side of the DEX and to incur a large price impact. The additional DEX liquidity weakens the price impact of clustered informed trading and attracts more informed traders to the DEX. On the other hand, the reaction of liquidity traders to the additional liquidity is weaker than that of informed traders. This is because their trading behavior stems from exogenous random reasons, and each liquidity trader expects that random buy and sell orders by other liquidity traders tend to cancel out each other on the DEX. It results in a small expected price impact, and deeper liquidity on the DEX has only a limited impact on liquidity traders' behavior. As a result, a more liquid DEX causes a larger migration of informed traders from the CEX to the DEX than that of liquidity traders, leading to a less severe adverse selection problem, a narrower bid-ask spread, and deeper liquidity on the CEX (Glosten and Milgrom, 1985).

Moreover, we formulate the expected profit function for liquidity providers on the DEX with asymmetric information. The existing theories of AMMs (e.g., Angeris and Chitra, 2020) argue that liquidity provision via AMMs only suffers from a cost, called *impermanent loss*, and a platform needs to pay some fees for liquidity providers to motivate them. The impermanent loss emanates from asymmetric information, as an informed trader imposes an adverse selection cost on liquidity providers.

In contrast to the literature, however, our model shows that liquidity providers also face lucrative trading opportunities even without fees. In particular, the expected value of liquidity pools improves when a trade is initiated by an *uninformed* liquidity trader. The profit opportunity is hard-wired in the convexity of the AMMs' pricing algorithm: when the liquidity pools randomly fluctuate along the convex curve, their expected value improves due to the Jensen's inequality. Incorporating the above profit and cost, we endogenize equilibrium liquidity on both exchanges. As in limit-order markets, liquidity on an automated market is negatively affected by the activity of informed traders relative to uninformed liquidity traders.

Several empirical implications are derived from our results. Since CFMMs are a class of convex pricing, consuming liquidity involves a larger price impact than adding liquidity.[4] In other words, given the trading size, a "buy" order tends to be more costly than a "sell" order. Due to this asymmetric price impact on the DEX, bid and ask prices on the CEX also tend to be asymmetrically distributed around the expected value of the asset. The asymmetric bid and ask prices are well documented in the literature (e.g., Ho and Stoll, 1981; Bossaerts and Hillion, 1991), and our model proposes the advent of CFMMs as a new source of asymmetric bid and ask prices. Also, our model suggests that a "sell" order flow tends to be more informative than a "buy" order flow on the DEX when the asset

[4] For example, with the CPMM, the execution price is determined so that the liquidity pools (x, y) shift along the convex curve, $y = \frac{k}{x}$. Thus a positive shift in x (a sell order) requires a smaller adjustment in y than the case of a negative shift in x (a buy order).

volatility increases. This is because informed sellers and buyers exhibit asymmetric reactions to the asset volatility, while the asymmetric reaction of liquidity traders are relatively weak due to their limited expected price impact. These implications, as well as the negative impact of the advent of DEXs on the CEX's bid-ask spread, can be tested by analyzing the listing of new cryptocurrency/token pairs on a DEX, such as Uniswap. There are several ERC-20 tokens pegged to some heavily traded cryptocurrencies. For example, Wrapped Bitcoin (WBTC) and Wrapped Ethereum (WETH) are one of the ERC-20 tokens pegged to Bitcoin and Ethereum, both listed on Uniswap. We expect to observe a shrink in the bid-ask spread of BTC/ETH exchange rate on centralized exchanges (e.g., Coinbase) when Uniswap announced that it starts to trade WBTC/WETH.

This paper is built on the large body of literature on market microstructure. In particular, Glosten and Milgrom (1985) and Kyle (1985) provide models of market liquidity with asymmetric information. Following the idea of Bagehot (1971), they show that competitive liquidity providers try to countervail the adverse selection cost of informed trading by making a market less liquid (i.e., posting a wider bid-ask spread; increasing the price impact of order flows). This paper applies their canonical idea to the new context of decentralized exchanges. We show that adverse selection still plays a key role in explaining liquidity provision in a market with AMMs.

Also, the modern financial market has experienced substantial fragmentation of trading exchanges, and several papers have addressed implications of coexisting exchange platforms with different market microstructures. For example, Ye (2011), Zhu (2014), and Ye (2016) consider the addition of so-called dark pools and analyze the reaction of informed and uninformed traders, as well as market quality. Lee (2019) investigates traders' behavior when multiple exchanges have different degrees of latency and transparency. More recently, exchanges start imposing "speed bumps" to slow down high-frequency trading and protect market makers against latency arbitrage. Brolley and Cimon (2020) analyze the order flow segmentation with speed bumps to address the liquidity impact of the slow market structure. Our model also sheds light on the liquidity impact of heterogeneous market structures in the era of decentralization and blockchain by incorporating endogenous order flow segmentation.

Our model also contributes to the research on the blockchain, cryptocurrency, and decentralized exchanges. The literature is expanding (see Harvey, 2016, Chen, Cong and Xiao, 2019, and Harvey, Ramachandran and Santoro, 2020 for comprehensive reviews), and many papers have analyzed the blockchain protocol as a new method or a platform for value transfer, e.g., Chiu and Koeppl (2017), Malinova and Park (2017), Cong, Li and Wang (2018), Pagnotta and Buraschi (2018), Schilling and Uhlig (2018), Abadi and Brunnermeier (2018), Huberman, Leshno and Moallemi (2019), and Lehar and Parlour (2019). However, those studies either consider only order book markets or abstract away from the formal description of matching or pricing algorithm on decentralized platforms. Our model complements the literature by analyzing the equilibrium with the automated market-making system that coexists with the traditional order book mechanism.

Formal descriptions and implementational details of decentralized exchanges are provided by, for example, Warren and Bandeali (2017), Adams, Zinsmeister and Robinson (2020), and Zhang, Chen and Park (2018). Although the analyses on AMMs is in its infancy, Angeris, Kao, Chiang, Noyes and Chitra (2019) provide a formal model of the optimal arbitrage problem with constant product market makers. Also, Angeris and Chitra (2020), Evans (2020), and Angeris, Evans and Chitra (2020) analyze more general CFMMs. Our model generalizes the above studies by incorporating asymmetric information between traders and traders' endogenous venue choice over coexisting markets with different market-making algorithms.

2 Technology overview

This section is devoted to preliminary discussions. We briefly describe trade execution on decentralized exchange platforms operated with the CPMM, a leading example of CFMM (or, in general, AMMs). Appendix A provides overview of the blockchain technology.[5]

2.1 Decentralized exchanges

Building a trading platform on the blockchain—i.e., a decentralized exchange—looks a natural strategy to extricate financial trading from a centralized information management and to make it robust to cyber attacks or single point of failures. However, maintaining a limit order book by a smart contract on the Ethereum blockchain is costly and tends to be slow, due to the time-consuming mining process, a complicated matching mechanism of limit order books, and the limited capacity of the blockchain.

[5] Readers can refer to Antonopoulos (2014) and Antonopoulos and Wood (2018) for more details on the blockchain technology.

There are two major solutions: a hybrid system and the Automated Market Makers (AMM). Many DEXs have adopted some "hybrid" mechanisms that involve both on-chain and off-chain features. For example, `0x` is built on so-called the *relayer mechanism* (see Warren and Bandeali, 2017). It provides an off-chain order book, on which traders can broadcast their trading intention and find their counterparties, as in the traditional centralized limit order markets. Since the order book is maintained off-chain, it refreshes swiftly. Once traders agree on a trade (i.e., trade execution), the order is settled on the blockchain via smart contracts. Note that the hybrid system still involves centralized protocol to a certain extent, as the relayers reserve some centralized power.

The second type of DEXs operate with AMMs. As mentioned in the introduction, it is a single-function algorithm that determines a price for order execution. As it is more simple than a limit-order matching mechanism, it requires much smaller computational capacity, making trade execution and settlement on the blockchain easier and faster.

2.2 Constant Product Market Makers

This section briefly explains how constant function market makers (CFMMs) determine the execution price of a trade by taking constant product market makers (CPMMs) as a leading example. It follows the specification by Zhang et al. (2018) and more detailed discussions are provided by, for example, Adams et al. (2020).

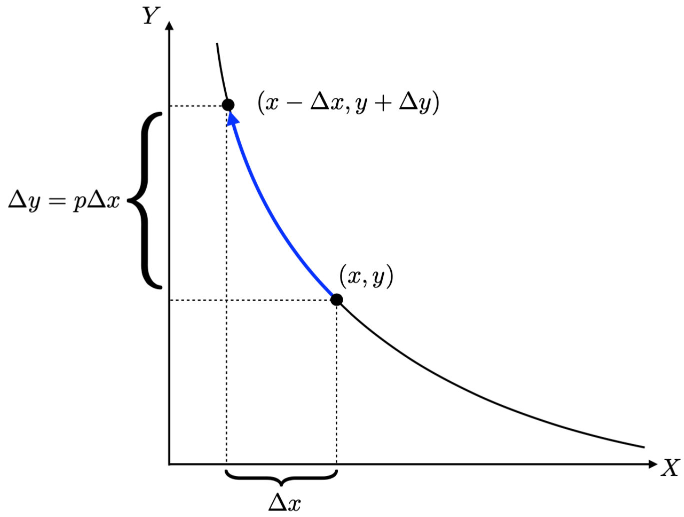
Liquidity pools and asset prices. Consider token X and token Y. Market makers inject tokens into an exchange following a certain rule described below. The exchange aggregates locked tokens and creates *a liquidity pool*. Suppose that the exchange reserves x unit of token X and y unit of token Y. The CPMM requires the geometric mean of the liquidity pools to be constant. That is, with some constant k, it must hold that $k = xy$.

If a trader wants to buy Δx of token X by selling $\Delta y = p\Delta x$ of token Y at price p, she adds Δy of token Y to the pool and withdraws Δx of token X from the pool. It triggers the following change in the pools:

$$x \to x' = x - \Delta x,$$

$$y \to y' = y + \Delta y.$$

Figure 2: Constant Product Market Makers



Note: This figure illustrates a change in the state of liquidity pools when an incoming market order is buying Δx unit of token X. The CPMM requires the liquidity pools to stay on the convex curve by adjusting a change in token Y or, equivalently, the execution price p.

Note that the price of token X in terms of token Y is $p = \frac{\Delta y}{\Delta x}$. Since the geometric mean of the pool must be constant, the price must satisfy the following equation.

$$k = x'y' = (x - \Delta x)(y + p\Delta x).$$

Thus, the above equation determines p as a function of the current state of the pool, (x, y), and the trading quantity Δx. In particular, we obtain

$$p = \frac{y}{x - \Delta x}.$$

Thus, the larger quantity the trader wants to buy ($\Delta x > 0$), the higher price she must pay, i.e., the price is an upward-sloping curve against the trading quantity. The price impact is mitigated when the exchange has a large amount of tokens in its liquidity pool.

Also by considering a small trading volume, $\Delta x \to 0$, the execution price for an infinitesimal trade is given by $p = y/x$, that is, the relative size of liquidity pools. Figure 2 shows a change in the pools' state caused by the above transaction: the exchange rate for an infinitesimal trade is determined by the slope of the curve specified by $k = xy$. Importantly, this implies that the price is convex in the trading volume.

Moreover, the pricing algorithm of the CPMM (or AMMs, in general) satisfies the property called *path independence*, i.e., when the liquidity pools move from one state to another state, the expected execution price is independent of the paths that the pools move. In the context of our paper, this means that trading a certain amount of assets all at once is equivalent to splitting orders and sequentially trading.[6] Appendix D provides a formal proof for this point.

Liquidity providers. When a market maker (or a liquidity provider) supplies liquidity via the CPMM, she is required to lock both token X and token Y. The amount of supplied liquidity must be adjusted following the current asset price on the DEX. For instance, suppose that the liquidity pools have x and y. Then, the price of token X in terms of token Y for an infinitesimal trade is $\frac{y}{x}$. If market maker i wants to supply x_i unit of token X, she must also supply the corresponding amount of token Y so that the slope of the CPMM does not change. This means that she also locks y_i such that $\frac{y+y_i}{x+x_i} = \frac{y}{x}$ or, equivalently, $y_i = \frac{y}{x}x_i$. According to this rule, market makers inject liquidity to the pools.

Although the geometric mean of the liquidity pools stays constant, a transaction causes a change in the liquidity pools, i.e., $(x, y) \to (x', y')$. The change in the value is the source of the profit and the cost of liquidity provision. In particular, if market maker i injects (x_i, y_i) before a trade, and the aggregate size of the pools is (x, y), she obtains the share of the pools, $w = \frac{y_i+x_i}{x+y}$. Once a trade is executed, the market maker can withdraw her share from the liquidity pools and realizes her returns. Since the post-trade pool state is (x', y'), the market maker obtains the gross return of $\pi = w(x' + y')$.

Constant Function Market Makers. A CFMM is a broader class of AMMs that is specified by a certain function $f : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ that maps the current state of the liquidity pools to some constant.[7] With the above example, the execution price p is determined by

$$f(x, y) = f(x - \Delta x, y + p\Delta x).$$

f must satisfy some regularity conditions so that $p \geq 0$ is uniquely determined by the above equation and well behaved (see Subsection 3.2 for formal analyses). Note that the CPMMs are the special case

[6] As discussed by Angeris et al. (2019), splitting orders costs more than trading all at once if a trader must pay a fee for trade execution.

[7] We assume that the automated market deals with exchanges of two assets but, in general, it can be defined with $n \geq 2$ assets by considering $f : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$.

of the CFMMs with $f(x, y) = xy$.

3 The model

Consider a one-shot trading game in a two-period economy. A single risky asset with common value $\tilde{v}$ is traded between three types of traders: for-profit traders, liquidity traders, and market makers. Throughout the paper, we use the term "asset" to underscore the model's broader generality, and cash (or a fiat currency) serves as numeraire. Alternatively, we can think of the asset as a digital token (e.g., some ERC-20 token) and cash as another token where $\tilde{v}$ is their relative value. With this interpretation, the asset price is an exchange rate between tokens.

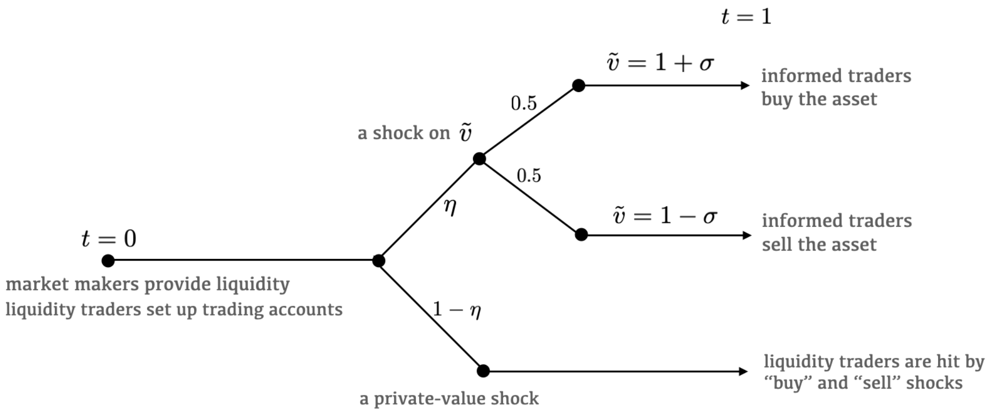
Events and traders. One of two possible event types triggers a trade at $t = 1$: either an innovation in the fundamental value of the asset (a common-value shock) or a liquidity shock (a private-value shock).[8] With probability $\eta \in (0, 1)$, the common value of the asset experiences an innovation and becomes $\tilde{v} = v_0(1 + \tilde{\sigma})$, where $\tilde{\sigma}$ is the stochastic growth rate of the asset's value and $\tilde{\sigma} = \pm\sigma$ with the same probability. v_0 represents the prior expected value of the asset and, without loss of generality, we normalize $v_0 = 1$.

There is a continuum of risk-neutral *for-profit traders* with a unit measure. When a common-value shock hits $\tilde{v}$, they immediately observe the realized value of the shock and choose their trading venue (defined below). A for-profit trader sends a single-unit market order to take an arbitrage opportunity. They represent sophisticated institutional investors in the real financial market and are referred to as *informed traders*.

With probability $1 - \eta$, a shock hits the private value of *liquidity traders*. Liquidity traders are impatient investors with no material information. A private-value shock triggers their needs for immediacy, such as hedging motives, margin constraints, and other immediate borrowing and lending requirements, exogenously making them want to trade. We assume that mass z_{buy} of liquidity traders are hit by a positive shock and buy one unit of the asset by sending a market order, whereas mass z_{sell} of liquidity traders are hit by a negative shock and try to sell one unit of the asset. The mass of liquidity traders is stochastic and uniformly distributed, $z_i \sim U[0, z]$ for $i \in \{buy, sell\}$. Upon

[8] See, for example, Menkveld and Zoican (2017) and Brolley and Zoican (2020) for models with these shocks as a trigger of transactions.

Figure 3: Timeline of the game



arriving at the market, each liquidity trader immediately places a single-unit market order to fulfill her trading needs.

In the following section, we first assume that the liquidity traders need to decide on where to place their order at $t = 0$ before they enter the market. This is because they are unsophisticated retail investors, and maintaining multiple accounts at both exchanges (or subscribing to a smart order router) is costly for them.[9] In Appendix B we relax this assumption and allow liquidity traders to choose trading venues contingent on the sign of a private-value shock.

There also exist a continuum of uninformed *market makers* with a sufficiently large measure. At the beginning of the game, competitively many market makers either post limit orders, each with a single unit, on a limit order market or locks one unit of assets in the liquidity pools of an automated market. Their behavior will be formally defined below.

Figure 3 illustrates the timeline of the game and possible outcomes of the trigger event.

Exchange platforms. There are two exchange platforms: a CEX and a DEX. The CEX is a traditional centralized exchange and operated with a continuous limit order book (LOB). Market makers on the CEX competitively provide quotes by submitting a single-unit limit order with bid and ask prices, as in Glosten and Milgrom (1985). The CEX is based on the centralized matching algorithm with high-speed information processors. Thus, it provides ultra-fast trade execution, causing almost no

[9] As of March 2021, there exist only a limited number of cryptocurrency exchanges that provide order routine services across CEXs and DEXs. An investor may trade via institutional brokers, but a large portion of cryptocurrency trades are directly done by retail investors.

delays.[10]

In contrast, the DEX is operated with the CFMM. As explained in Subsection 2.2, market makers competitively lock one unit of the asset (and the corresponding value of cash) into the exchange, generating *liquidity pools*. A liquidity taker who wants to buy (resp. sell) the asset subtracts (resp. adds) the asset from the asset liquidity pool by adding (resp. subtracting) cash to the cash liquidity pool. The execution price is determined by the CFMM algorithm instead of quotes by market makers.

Trading with the CFMM involves smart contracts on Ethereum blockchain, and it takes much lower throughput than the CEX, causing a delay in completion of a transaction. We assume that all incoming orders placed at $t = 1$ are simultaneously processed at the end of the first period. However, transactions are stored in the mempool of the blockchain and wait for validation and execution by blockchain miners. Following Zhu (2014), a delay in transactions weighs negatively on the private utility of liquidity traders, as they are impatient and eager to fulfill their trading needs immediately. In the model, a liquidity trader on the DEX incurs $\gamma\sigma$ of delay costs per unit of trade, where γ is a random parameter that measures the aversion toward a delay (or needs for immediacy) and drawn from $\gamma \sim U[0, 1]$.[11]

Differentiation. The informed traders buy (resp. sell) the asset when the asset value experiences a positive jump $\tilde{\sigma} = +\sigma$ (resp. a negative jump, $\tilde{\sigma} = -\sigma$). It becomes clear in the following discussion that the behavior of informed traders is asymmetric and depends on the trade direction due to the convex nature of the execution price on the DEX. Therefore, conditional on $\tilde{\sigma} = +\sigma$, $\beta_{buy} \in [0, 1]$ fraction of informed traders participate in the DEX to buy the asset. In contrast, with $\tilde{\sigma} = -\sigma$, measure $\beta_{sell} \in [0, 1]$ of informed traders sell on the DEX, where we allow $\beta_{buy} \neq \beta_{sell}$. We denote the measure of liquidity traders on the DEX as $\alpha \in [0, 1]$, which is not contingent on the sign of a private-value shock, as they decide on trading venues at $t = 0$ (see Appendix B for the case with asymmetric α).

[10]From the high-frequency traders' perspective, even a centralized exchange with cutting-edge technologies causes a microsecond or nanosecond delay that may affect trading profits. However, the primary focus of this paper is on the trading on the CEX compared to that on the DEX, and ignoring delays in order of microseconds or nanoseconds does not harm our discussions, as a delay on the DEX is the order of seconds or minutes, which is much longer than that on the CEX.

[11]The delay cost that is proportional to the asset volatility σ can be seen as margin constraint or unmodeled risk aversion (e.g., Brunnermeier and Pedersen, 2009).

3.1 Trading on the CEX

The partial equilibrium on the CEX with the limit order book is quite standard and follows the model by Zhu (2014). We denote the equilibrium bid and ask prices on the CEX as

$$\text{Ask} = 1 + a, \ \text{Bid} = 1 - b.$$

We sometimes call the deviation of Ask and Bid from the expected value of the asset the ask (a) and bid ($-b$) prices. Also, the (effective) bid-ask spread is defined as $S = a + b$.

Given the differentiation of traders, the expected profits for a market maker on each side of the market are given by

$$\pi^{C}_{M,ask} = \frac{1}{2}\left[\eta(1-\beta_{buy})(a-\sigma) + (1-\eta)(1-\alpha)za\right],$$
$$\pi^{C}_{M,bid} = \frac{1}{2}\left[\eta(1-\beta_{sell})(b-\sigma) + (1-\eta)(1-\alpha)zb\right].$$

In both equations, the first term represents a trade with an informed trader, which happens with expected amount $\frac{1}{2}\eta(1-\beta_i)$, and the second term shows the case of liquidity trading, which happens with expected amount $\frac{1}{2}(1-\eta)(1-\alpha)z$. Following Zhu (2014), we focus on the equilibrium in which a market maker breaks even on the both sides of the market. Then, the break-even condition yields the following competitive bid and ask prices.

$$a = a(\beta_{buy}, \alpha) = \sigma\frac{(1-\beta_{buy})\eta}{(1-\beta_{buy})\eta + (1-\eta)(1-\alpha)z}, \tag{1}$$
$$b = b(\beta_{sell}, \alpha) = \sigma\frac{(1-\beta_{sell})\eta}{(1-\beta_{sell})\eta + (1-\eta)(1-\alpha)z}. \tag{2}$$

Note that the above prices are potentially asymmetric, as we allow $\beta_{buy} \neq \beta_{sell}$. Otherwise, the comparative statics of the bid-ask spread are the same as the traditional models of market microstructure with asymmetric information. That is, the bid-ask spread is positively (resp. negatively) affected by the intensity of informed (resp. liquidity) trading, as it exacerbates (resp. mitigates) the adverse selection cost for a market maker.

Accordingly, the expected profits for an informed trader who trades on the CEX are given by

$$\pi_I^C(\tilde{\sigma}) = \begin{cases} \sigma - a & \text{if } \tilde{\sigma} = +\sigma \text{ and buys the asset,} \\ \sigma - b & \text{if } \tilde{\sigma} = -\sigma \text{ and sells the asset.} \end{cases} \tag{3}$$

Note that the informed trader's profits are computed conditional on the realized value of $\tilde{\sigma}$. Similarly, a liquidity trader's (ex-post) profits from trading on the CEX are given by[12]

$$\pi_{L,k}^C = \begin{cases} -a & \text{if } k = buy, \\ -b & \text{if } k = sell, \end{cases} \tag{4}$$

where subscript k indicates whether the private-value shock induces a trader to buy or sell the asset. Since z_{buy} and z_{sell} measures of liquidity traders are hit by "buy" and "sell" private-value shocks, the aggreagate ex-ante expected trading cost for a liquidity trader at $t = 0$ is equivalent to the effective bid-ask spread:

$$\mathbb{E}\left[\sum_{k=buy,sell} z_k \pi_{L,k}^C\right] = -\frac{z}{2}S. \tag{5}$$

3.2 Trading on the DEX

Pricing by Constant Function Market Makers. For this section, we assume that the liquidity pools have an exogenous amount of cash and the asset, denoted as C and X, while Section 4 considers endogenous C and X.

Suppose that an incoming market order tries to trade δ unit of the asset (e.g., $\delta > 0$ means a "buy" market order with mass δ). By denoting the execution price for the order as p, the state of the liquidity pools changes as follows after the trade:

$$C \to C' = C + p\delta \tag{6}$$

$$X \to X' = X - \delta. \tag{7}$$

[12]We implicitly assume that a liquidity trader obtains private utility u if she fulfills her trading needs, with u being sufficiently large (e.g., $u > 1 + \sigma$). Therefore, all liquidity traders participate in the market upon hit by a shock. u does not affect the equilibrium conditions because a liquidity trader obtains u no matter where she trades.

For example, if a trader is buying the asset ($\delta > 0$), she subtracts δ unit of the asset from the asset liquidity pool, X, by adding the corresponding amount of cash $p\delta$ to the cash pool, C.

In our model, the CFMM is defined by function $f : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ that determines the state of the liquidity pools, (c, x), and pricing function p derived from f. Instead of enumerating the regularity conditions for f, Lemma 1 below presents the conditions that p satisfies, as they have clearer intuition. Appendix C provides the conditions for f and demonstrates that they lead to those for p.

The CFMM sets the execution price p so that the post-trade state of the pools satisfies

$$f(C, X) = f(C + p\delta, X - \delta).$$

The regularity conditions for f guarantee that p is uniquely determined as a function of δ and (C, X). In the following, we introduce several functions that simplify the discussions.

Firstly, with the regularity conditions, we assume that there is no arbitrage at $t = 1$.

Assumption 1. *At the beginning of the trading game, there is no arbitrage, and an infinitesimal trade cannot make a strictly positive profit.*

Note that an infinitesimal trade ($\delta \to 0$) is executed at price

$$p_0 \equiv \frac{f_x(C, X)}{f_c(C, X)}, \tag{8}$$

where we denote the partial derivative of $f(c, x)$ with respect to $j = c, x$ as f_j. Therefore, the non-arbitrage condition implies that $p_0 = \mathbb{E}[\tilde{v}] = 1$, and equation (8) pins down the relationship that the initial liquidity pools, (C, X), must satisfy. In particular, there exists a differentiable and increasing function, denoted as $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$, such that

$$C = g(X).$$

As explained in Subsection 2.2, the above equation determines the rule for liquidity providers when we endogenize (C, X) in Section 4.

Secondly, we characterize a set of reachable states, (c, x), on the CFMM curve. Namely, if state (c, x) is on the CFMM curve with the initial condition (C, X), the value of c is expressed by using monotonically decreasing function $h : \mathbb{R}_{++} \to \mathbb{R}_{++}$ as

$$c = h(x; C, X). \tag{9}$$

By using the above functions, we can translate the regularity conditions in Appendix C into the pricing equation. Proofs for the following results are provided in Appendix C.

Lemma 1. *Given the initial state of the liquidity pools $(g(X), X)$ and function h defined by the CFMM (9), the execution price for a trade with size $\delta \neq 0$ is given by*

$$p(\delta, X) = \frac{1}{\delta} \int_0^\delta \frac{f_x(h(X - \tilde{\delta}), X - \tilde{\delta})}{f_c(h(X - \tilde{\delta}), X - \tilde{\delta})} d\tilde{\delta}. \tag{10}$$

Price function p satisfies the following conditions.

(i) $p_0 = p(0, X) = 1$;

(ii) p is increasing in δ;

(iii) p is decreasing in X if and only if $\delta > 0$;

(iv) p is differentiable with respect to both elements, and $\frac{\partial^2 p}{\partial \delta \partial X} < 0$;

(v) $p_\delta(b, X) > p_\delta(-b, X)$ holds for all $b, X > 0$;

(vi) $A(\delta, X) \equiv |p(\delta, X) - 1|$ is log-submodular in (δ, X) for $\delta \neq 0$.

Given the initial liquidity X, conditions in Lemma 1 only requires monotonicity and convexity of the price function with respect to trade size δ. They imply that $p(\delta, X)$ has typical properties as a price function. Firstly, condition (i) is the repetition of the non-arbitrage condition. Secondly, the larger quantity the trader intends to buy (resp. sell), the higher (resp. lower) the execution price becomes (condition [ii]). We can also check that ample liquidity lowers the price level given the size of a trade (condition [iii]).

Moreover, we can define the price impact of a trade, and condition (iv) defines its behavior.

Lemma 2. *The price impact of a market order with measure δ is defined by $\kappa(\delta, X) = \frac{d \log p(\delta, X)}{dx}$. κ is decreasing in X and increasing in δ.*

The above lemma characterizes the market depth on the DEX. When the DEX has a larger quantity in its liquidity pools (X), the market becomes deeper, and a market order of a given size has a smaller price impact. Therefore, Lemma 2 suggests that it is appropriate to use the pool size (X) as the measure of liquidity on the DEX.

Examples. When an automated market is operated by the CPMM, we have $f(C, X) = CX$. The attainable states on f is characterized by $c = h(x; C, X) = \frac{XC}{x}$. Also, the pricing algorithm is given by

$$p(\delta, X) = \frac{X}{X - \delta}$$

where we applied the non-arbitrage condition to obtain $C = g(X) = X$ for the initial condition. The price impact for a trade with size δ is $\kappa = \frac{1}{X-\delta}$, which is increasing in δ and decreasing in X. Also, p is a convex function of δ.

One of the other popular CFMMs is the Constant Mean Market Makers (CMMMs) defined by

$$f(C, X) = C^w X^{1-w}$$

with some parameter $w \in (0, 1)$. It holds that $c = h(x; C, X) = C\left(\frac{X}{x}\right)^{\frac{1-w}{w}}$, and the initial condition becomes $C = g(X) = \frac{w}{1-w}X$. The price is given by

$$p(\delta, X) = \frac{w}{1-w} X^{\frac{1}{w}} \frac{(X-\delta)^{\frac{w-1}{w}} - X^{\frac{w-1}{w}}}{\delta}.$$

It is easy to check that the above price satisfies all conditions in Lemmas 1 and 2.

Profits of informed traders on the DEX. Now, we go back to the general CFMM and use p in equation (10). For brevity, we omit X in the price function and denote it as $p(\delta)$.

When a jump in the asset's value occurs, each informed trader knows $\tilde{\sigma} = \pm\sigma$, which also implies that she is aware of the trading attempts of other informed traders. More precisely, she knows that a positive (resp. negative) jump triggers buy (resp. sell) market orders of aggregate size β_{buy} (resp. β_{sell}). Therefore, conditional on $\tilde{\sigma} = \pm\sigma$, the execution price on the DEX is given by

$$p = \begin{cases} p(\beta_{buy}) & \text{when } \tilde{\sigma} = +\sigma, \\ p(-\beta_{sell}) & \text{when } \tilde{\sigma} = -\sigma. \end{cases} \tag{11}$$

For simplicity, we assume that all incoming orders are aggregated and executed all at once, but Appendix D shows that the expected execution price is the same as (11) when we consider uniform arrivals and sequential execution of orders due to the *path independence*. As a result of (11), the informed trader's expected profit from trading on the DEX is

$$\pi_I^D(\tilde{\sigma}) = \begin{cases} 1 + \sigma - p(\beta_{buy}) & \text{if } \tilde{\sigma} = +\sigma \\ p(-\beta_{sell}) - (1 - \sigma) & \text{if } \tilde{\sigma} = -\sigma. \end{cases} \tag{12}$$

We can think of $p(\delta)$ and $p(-\delta)$ as the ask and the bid prices on the DEX but $p(\delta) - 1 \neq 1 - p(-\delta)$ as long as $\delta \neq 0$. This is because of the convexity in the CFMM's pricing algorithm, stipulated in Lemma 1.

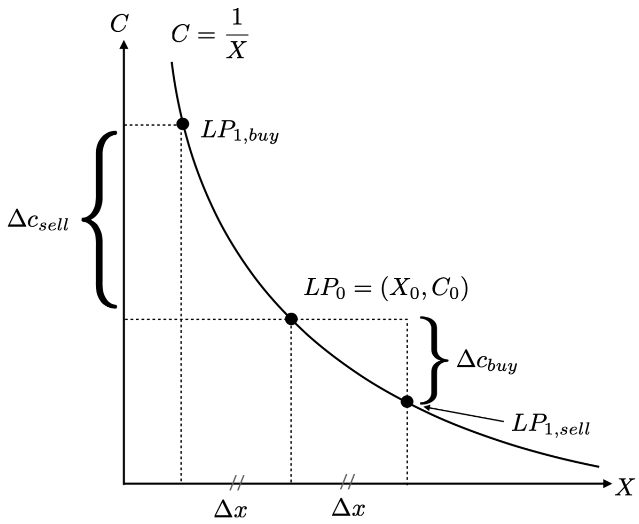
Figure 4 illustrates intuition by taking the CPMM as an example. Suppose that a buy or a sell order with quantity $\delta = \Delta x$ moves the state of the liquidity pools from LP_0 (the initial condition) to LP_1 in the figure. If it is a buy order ($\Delta x > 0$), LP_0 slides upward to $LP_{1,buy}$, while a sell order moves LP_0 downward to $LP_{1,sell}$, both along the curve $C = X^{-1}$. Since the curve is convex, even if the buy and the sell orders have the same size, the buy order requires a larger adjustment of the cash pool than the sell order, $\Delta c_{buy} > \Delta c_{sell}$. Hence, a trader must pay a higher price p to buy the asset than the price payment she obtains when she sells the asset.[13]

The above discussion is true even if we switch the role of cash and the asset. Economically, the convexity of the CFMM implies that the execution price or the exchange rate between assets is determined so that adding the liquidity to the pool bears a smaller price impact compared to consuming liquidity in the pool. This hard-wired asymmetry is absent in the limit order market because, without frictions or risk aversion, a buy and a sell market order bear the same trading cost, i.e., a bid and ask prices are symmetric around the mid point.

Profits for liquidity traders on the DEX. With probability $1 - \eta$, the trigger event is a private-value shock on the needs for immediacy of liquidity traders. Conditional on the direction of the private-

[13] Although the curve $y = k/x$ is symmetric around $x = y$, the reaction of y is asymmetric to $x \pm d$ as long as $d > 0$.

Figure 4: Asymmetric price impact



Note: This figure discribes the price impact of buy and sell orders with the same size, Δx. Starting from (X_0, C_0), a buy market order with size Δx causes an upward shift of the liquidity pools, while a sell order triggers a downward shift. The execution price must keep (C, X) on the CPMM curve ($C = 1/X$). Since the curve is convex, an addition of Δx to X requires a smaller adjustment of the value of C than the case of reduction of X, meaning that the price impact is larger for a buy order.

value shock, the profits for a liquidity trader with the delay cost γ are given by

$$\pi^D_{L,k}(\gamma) = \begin{cases} 1 - \mathbb{E}_{\Delta z}[p(\alpha\Delta z)] - \gamma\sigma & \text{if } k = buy, \\ \mathbb{E}_{\Delta z}[p(\alpha\Delta z)] - 1 - \gamma\sigma & \text{if } k = sell. \end{cases} \tag{13}$$

The aggregate order size, $\Delta z = z_{buy} - z_{sell}$, is uncertain for each liquidity trader, and $\mathbb{E}_{\Delta z}$ is the expectation over Δz. Compared to the informed trading, the price impact of liquidity trading tends to be weak in expectation because their trading direction is random, and buy and sell orders are netted out.

3.3 Equilibrium venue choice

We define the equilibrium with exogenous DEX liquidity (C, X) as follows:

Definition 1. *The equilibrium of the model is defined by the measure of informed and liquidity traders who participate in the DEX, $\{(\beta_i)_{i=buy,sell}, \alpha\}$, the bid-ask prices, (a, b), and the execution prices on the DEX, p, such that (i) the informed traders are indifferent between trading on the CEX and the DEX given the occurrence of a jump in $\tilde{v}$ and the information about $\tilde{\sigma}$, (ii) the liquidity traders are differentiated by comparing the ex-ante* *expected profits on the CEX and the DEX, (iii) the market makers on the CEX break even at the bid and the ask prices, and (iv) the prices on the DEX follows the CFMM algorithm given the initial pools condition (C, X).*

With the profit functions for each trader on each venue given by (3), (4), (12), and (13), the indifference conditions for traders pin down the equilibrium mass of traders on the DEX.

Informed traders. Firstly, the measure of informed traders on each exchange is characterized by the following.

Proposition 1. *(i) Given α, the equilibrium measures of informed traders on the DEX, β^*_{buy} and β^*_{sell}, are the solution of the following equations, respectively:*

$$1 + \sigma \frac{(1-\beta)\eta}{(1-\beta)\eta + (1-\eta)(1-\alpha)z} = p(\beta),$$
$$1 - \sigma \frac{(1-\beta)\eta}{(1-\beta)\eta + (1-\eta)(1-\alpha)z} = p(-\beta).$$

The solutions exist and are unique.
(ii) β^*_i is monotonically increasing in α, X, C, and σ.
(iii) $\beta^*_{buy} < \beta^*_{sell}$ for all $\alpha \in (0, 1)$.

β^*_{buy} and β^*_{sell} are both a unique solution for each indifference condition in $\beta \in (0, 1)$, and they are stable, i.e., they do not diverge even if a small perturbation happens to parameter values. Intuitively, an increase in the measure of DEX informed traders β^* causes a larger price impact on the DEX and a narrower bid-ask spread on the CEX. Thus, the CEX becomes more attractive than the DEX, meaning that a marginal informed trader on the CEX has no incentive to switch her trading venue.

Moreover, the trading intensity of the informed traders exhibits anticipated reactions to a change in parameter values. If a larger set of liquidity traders participate in the DEX (α increases), it exacerbates adverse selection for market makers on the CEX. It widens the bid-ask spread, and more informed traders are willing to participate in the DEX (β^*_i increases). Also, a larger X (and thus C) attenuates the price impact of informed trading (see Lemma 2), inviting more informed traders to the DEX. Finally, a higher volatility of the asset (σ) also attracts informed traders to the DEX, as the bid-ask spread on the CEX becomes proportionally wider when σ increases, while the execution price at the DEX is not directly affected.

One of the novel features of the CFMM algorithm emanates from asymmetry in execution prices due to its convexity explained above. Since buying the asset incurs a larger cost than the return of selling the asset, a negative innovation in the asset's value induces a disproportional reaction of informed sellers to informed buyers, leading to $\beta^*_{buy} < \beta^*_{sell}$. This asymmetry provides some empirical implications, as discussed later.

Liquidity traders. A liquidity trader with cost parameter γ participates in the DEX if and only if the profits from trading on the DEX dominate that from trading on the CEX:[14]

$$\frac{z}{2}\sum_{k=buy,sell}\pi^D_{L,k} \geq \frac{z}{2}\sum_{k=buy,sell}\pi^C_{L,k} = -\frac{z}{2}S,$$

where the last equality uses (5). The above inequality is reduced to

$$\gamma < \gamma^* \equiv \frac{S(\beta_{buy}, \beta_{sell}, \alpha)}{2\sigma}.$$

The LHS is the expected trading cost on the DEX. Since a liquidity trader buys or sells with the same probability, the execution price does not affect the expected cost. However, the delay cost matters because a liquidity trader on the DEX bears it regardless of the trading direction. In contrast, the RHS represents the expected trading cost on the CEX, i.e., the bid-ask spread. Since $\gamma \sim U[0,1]$, we obtain the following result.

Proposition 2. *(i) Given $(\beta_{buy}, \beta_{sell})$, the equilibrium mass of liquidity traders on the DEX is determined by*

$$\alpha = \Pr(\gamma < \gamma^*) = \frac{S(\beta_{buy}, \beta_{sell}, \alpha)}{2\sigma}. \tag{14}$$

If $z > z^*$, equation (14) has two solutions: $\hat{\alpha} = 1$ and $\alpha^* \in (0,1)$. $\hat{\alpha}$ is unstable and α^* is stable. If $z < z^*$, $\hat{\alpha} = 1$ is a unique (unstable) solution. The threshold value z^*satisfies :

$$z^* = \frac{2\eta}{1-\eta}\frac{(1-\beta_{buy})(1-\beta_{sell})}{2-\beta_{buy}-\beta_{sell}}.$$

(ii) α^* is monotonically decreasing in β_{buy} and β_{sell}.

[14] We assume that a liquidity trader participates in the DEX when she is indifferent.

In what follows, we assume that the condition holds for multiple solutions and focus on the stable solution.

The measure of liquidity traders on the DEX is decreasing in the measure of informed traders on the DEX. When informed traders migrate away from the CEX to the DEX (β_i^* increases), the bid-ask spread on the CEX tightens. Since the execution price on the DEX for liquidity trading is not directly affected by the measure of informed traders, the CEX becomes more attractive for the liquidity traders, and α^* declines.

3.4 Liquidity implications of the CFMM

It is not trivial whether an additional liquidity on the DEX improves or deteriorates liquidity on the CEX. Since the bid-ask spread on the CEX is determined by the ratio of informed trading to liquidity trading, we need to investigate the reaction of $(\beta_{buy}, \beta_{sell})$ relative to that of α to a change in DEX liquidity, X.

By putting Propositions 1 and 2 together, we obtain the following result.

Proposition 3. *(i) The equilibrium measures of liquidity traders and informed traders on the DEX, $(\alpha^*, \beta_{buy}^*, \beta_{sell}^*)$, solve the following equations.*

$$\alpha = \frac{S(\beta_{buy}, \beta_{sell}, \alpha)}{2\sigma}, \tag{15}$$

$$\beta_i = \begin{cases} \beta_{buy}^*(\alpha) & \text{for } i = buy, \\ \beta_{sell}^*(\alpha) & \text{for } i = sell, \end{cases}$$

where β^* is given by Proposition 1. There is a unique stable interior solution for the above equations.

(ii) The equilibrium measure of liquidity traders on the DEX α^* is a decreasing function of DEX liquidity X.

(iii) The equilibrium measure of informed traders who buy on DEX β_{buy}^* is an increasing function of X. Moreover, the expected measure of informed traders on DEX in equilibrium $\frac{\beta_{buy}^*+\beta_{sell}^*}{2}$ is increasing in X.

(iv) The equilibrium bid-ask spread on the CEX is a decreasing function of X.

Proposition 3 shows that informed traders are more inclined to participate in the DEX when it becomes more liquid, whereas liquidity traders tend to trade on the CEX.

Firstly, informed traders are concerned about the price impact on the DEX, and it is decreasing in X. Thus, larger liquidity pools render informed trading on the DEX less costly and attract more informed traders to the DEX. In turn, more active informed traders on the DEX mitigate the adverse selection cost for CEX market makers, and the bid-ask spread declines.

Secondly, a change in X does not have a direct impact on liquidity traders' behavior, as the execution price on the DEX does not matter in expectation. Hence, facing a narrower bid-ask spread on the CEX, more liquidity traders participate in the CEX.

This process involves a decline in the bid-ask spread or improved market liquidity on the CEX, as demonstrated by point (iv) in Proposition 3. Therefore, our model suggests that liquidity on the DEX complements that on the CEX.

Remark. Although we do not model underling blockchain mechanisms for the DEX in detail, some exogenous variations in blockchain parameters may cause a change in DEX liquidity and affect CEX liquidity. For example, the amount of Ethereum locked in Uniswap has experienced a substantial drop (about 40%) in November 2020 after the announcement of a change in the platform's parameter.[15] The above result can be used as a simple building block to analyze the liquidity impact of such an event and helps discuss the broader implications of blockchain environment by incorporating its cross-market effects.

4 Liquidity provision with the CFMM

This section considers liquidity provision by market makers on the automated market and analyzes how X is determined and affected by traders' behavior.

4.1 Market makers' profits on the DEX

Prior to the trading game (at $t = 0$), each market maker decides on whether to supply liquidity. Since the market-making sector on the CEX is competitive and yields the zero profit in expectation, it works as an outside option for market makers on the DEX.

If a market maker provides liquidity on the DEX, she injects one unit of the asset and $c = g(X + 1) - g(X)$ unit of cash to the liquidity pools. She must follow this rule so that the non-arbitrage condition remains true. By locking liquidity with value $c + 1$ into the DEX, the market maker obtains $w = \frac{c+1}{C+X}$ share of the aggregate liquidity pools. With a trade of size δ, the post-trade liquidity pools

[15] See, for example, https://cointelegraph.com/.

have C' and X' in equations (6) and (7). After a trade, the market maker earns w share of (C', X'), realizing the difference from the initial cost as her net profit.

Since the order type and the trigger event are uncertain for market makers at the liquidity provision stage, the expected liquidation value of the liquidity pools is

$$V_{LP} = \mathbb{E}\left[g(X) + p(\tilde{\delta})\tilde{\delta} + \tilde{v}(X - \tilde{\delta})\right]$$

where $\mathbb{E}$ is the expectation operator regarding the type of traders (i.e., informed or uninformed) captured by $\tilde{\delta}$ and the innovation in the value of the asset, if any.

The impermanent loss. With probability η, information-driven traders take liquidity where they are buying and selling with measure β^*_{buy} and β^*_{sell}, respectively, with the same probability. In this case, the expected profit for a market maker, net of the initial cost of injecting assets, is given by π^D_{IT} below.

$$\begin{aligned}\pi^D_{IT} &= \frac{w}{2}\left[\overbrace{g(X) + p(\beta^*_{buy})\beta^*_{buy} + (1+\sigma)(X - \beta^*_{buy})}^{\text{if } \tilde{\sigma} = +\sigma} + \overbrace{g(X) - p(-\beta^*_{sell})\beta^*_{sell} + (1-\sigma)(X + \beta^*_{sell})}^{\text{if } \tilde{\sigma} = -\sigma}\right] - (c+1) \\ &= \frac{w}{2}\left[\left(p(\beta^*_{buy}) - (1+\sigma)\right)\beta^*_{buy} + \left((1-\sigma) - p(-\beta^*_{sell})\right)\beta^*_{sell}\right].\end{aligned}$$

The first line represents the post-trade value of the liquidity pools, which is comprised of the cases with positive and negative shocks, net of the initial cost. It reduces to the second line and leads to the following result.

Proposition 4. *(i) When the trade is triggered by a common-value shock, the market maker's expected net profit on the DEX is negative, i.e.,* $\pi^D_{IT} < 0$. *(ii) Given X,* π^D_{IT} *is decreasing in* σ.

The negative profit from informed trading is called the *impermanent loss* (see, for example, Angeris and Chitra, 2020). The fact that liquidity is taken by an informed trader implies that the value of liquidity pools inevitably declines. This is because an informed trader always subtracts one of the more valuable assets from the liquidity pools by adding a less valuable asset. As a result, a market maker ends up having a larger amount of a less valuable asset by giving up a more valuable one. This result highlights the similarity of the CFMM to the market making on the limit order book, where informed trading involves adverse selection for market makers due to information asymmetry (e.g., Glosten and Milgrom, 1985; Kyle, 1985).

Moreover, the adverse selection cost for DEX market makers (π_{IT}^D) magnifies when σ increases. The more volatile the asset becomes, the more informational advantage the for-profit traders obtain by knowing $\tilde{\sigma} = \pm\sigma$. Thus, σ represents the degree of adverse selection and the cost of liquidity provision on the DEX.

Profits from noise. When a trade is triggered by a shock on the private value of liquidity traders, it causes noise trading, i.e., the traders' behavior is independent of the value of the asset. It results in the market orders with stochastic size $\Delta z = z_{buy} - z_{sell}$. We denote the pdf and the cdf of Δz as q and Q, respectively.[16]

The expected net profit of a market maker, when a private-value shock triggers a trade, is given by

$$\begin{aligned}\pi_{LT}^D &= w \int_{-z}^{z} \left[g(X) + p(\alpha\Delta z)\alpha\Delta z + (X - \alpha\Delta z)\right] dQ(\Delta z) - (c+1) \\ &= w \int_{-z}^{z} p(\alpha\Delta z)\alpha\Delta z dQ(\Delta z).\end{aligned}$$

Proposition 5. *(i) When the trade is triggered by a private-value shock, the market maker's expected net profit on the DEX is positive, i.e., $\pi_{LT}^D > 0$.*
(ii) π_{LT}^D is increasing in α and decreasing in X. Furthermore, given X, it is decreasing in σ.

As in the case of limit order markets, market makers on the DEX gain from trading with liquidity traders because uninformed liquidity trading improves the value of the liquidity pools. Firstly, since liquidity buy and sell orders are netted out and they are independent of $\tilde{\sigma}$, liquidity trading does not change the expected value of the asset pool ($\mathbb{E}[X'] = X - \alpha\mathbb{E}[\Delta z] = X$), leading to zero net profits from $X \to X'$. The strictly positive profits emanate from the liquidity pool of cash. Since the execution price adjusts the post-trade liquidity pools along with the convex curve $f(C, X)$, Jensen's inequality implies that

$$\mathbb{E}[p(\alpha\Delta z)\alpha\Delta z] > 0.$$

[16]Since $\Delta z = z_{buy} - z_{sell}$ with $z_i \sim U[0, z]$, the pdf of Δz is

$$q(\Delta z = u) = \begin{cases} 0 & \text{if } u \notin [-z, z], \\ \frac{1}{z^2}(z - |u|) & \text{if } u \in [-z, z]. \end{cases}$$

Therefore, the positive impact of liquidity trading is hard-wired in the CPMM's convex pricing algorithm and works as an implicit reward for liquidity providers on the DEX.

The profit from noise trading is magnified when the volatility of liquidity trading is large. For example, both a larger set of liquidity traders on the DEX (α) and a wider variation in liquidity trading (i.e., $Var(z_i)$) improve π_{LT}^D. In contrast, a larger liquidity size (X) diminishes a variation in the liquidity trading and reduces π_{LT}^D. These properties are easily derived from Jensen's inequality.

The profit mechanism in Proposition 5 is absent in the literature of automated market makers. The existing theory has analyzed how the price on an automated market converges to an exogenous reference price, where arbitragers play their role to facilitate the price's convergence. We introduce liquidity or noise traders following the microstructure literature (e.g., Grossman and Stiglitz, 1980; Black, 1986; De Long et al., 1990) and show that they play an important role to motivate liquidity provision even without fee rebates to market makers.

4.2 Equilibrium with endogenous DEX liquidity

The expected profit from providing liquidity on the DEX is

$$\pi_M^D(X) = \eta \pi_{IT}^D(X) + (1-\eta)\pi_{LT}^D(X). \tag{16}$$

To endogenize liquidity on the DEX, we focus on the equilibrium with a free entry condition for market makers. Moreover, we assume that there exist a set of passive liquidity providers on the DEX who provide some exogenous amount $x_{passive} > 1 + \frac{1+\sqrt{1+\sigma}}{\sigma}$. This assumption guarantees that the equilibrium is well defined, i.e., even if the active liquidity providers provide zero liquidity, the DEX has a sufficiently large pools compared to the potential size of liquidity-taking orders.

Proposition 6. *(i) Given β_i^* and α^*, the market maker's expected profit is decreasing in X for $\pi_M^D \geq 0$.*

(ii) Given β_i^* and α^*, there is a unique and stable $X = X^*$ such that $\pi_M^D(X) = 0$. that solves the break-even condition in (16).

(iii) X^* is increasing in α^* and weakly decreasing in β_i^*.

Given the behavior of traders, the expected profit for each market maker is monotonically decreasing in the amount of the asset locked (for $\pi_M^D \geq 0$). As a larger number of liquidity providers participate and supply liquidity, the individual profit dilutes more.

Due to the free entry condition and the zero expected profits on the CEX, the equilibrium $X = X^*$ is determined by the break-even condition, $\pi_M^D(X^*) = 0$, and Proposition 6 suggests that there is a unique and stable X^*. As long as $\pi_M^D > 0$, more liquidity providers participate in the DEX and pushes up the value of X, reducing π_M^D. This process continues until it holds that $\pi_M^D \leq 0$. In contrast, if $\pi_M^D < 0$, liquidity providers stop supplying liquidity, and X declines until $\pi_M^D \geq 0$ holds. Therefore, the equilibrium is determined by the break-even condition, $\pi_M^D = 0$, and it is stable.

Moreover, Proposition 6 implies that the size of the liquidity positively reacts to an exogenous change in α, whereas it declines when β_i^* increases. Intuition follows the traditional discussions on adverse selection: informed trading relative to liquidity trading makes it more costly for market makers on the DEX to supply liquidity.

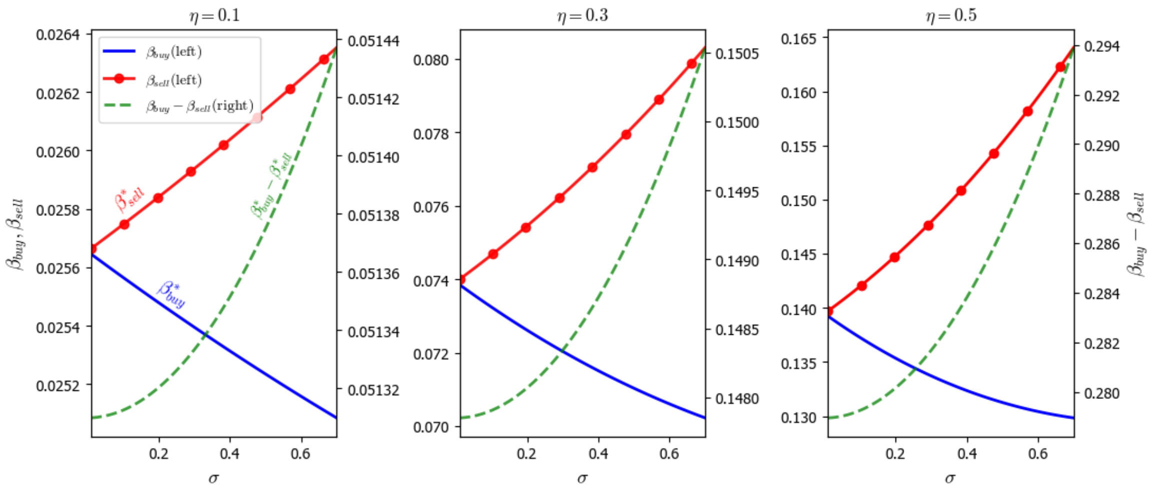
Comparative statics. We want to gauge a joint reaction of the traders' behavior and market liquidity to variations in an exogenous parameter. In what follows, we take the volatility of the asset σ (or the measure of asymmetric information) as a source of variations and numerically analyze the general equilibrium by taking the CPMM ($f(C, X) = CX$) as the leading example of CFMMs.

A higher volatility of the asset implies that informed traders possess a more informational advantage over market makers, and the adverse selection problem worsens both on the DEX and the CEX. Therefore, it confounds liquidity provision by DEX market makers, leading to a decline in X^*, as well as a wider effective bid-ask spread on the CEX. The above changes directly affect informed trading ($\beta_{buy}^*, \beta_{sell}^*$; via Proposition 1), whereas the measure of liquidity trading (α^*) is only indirectly affected by σ.

As both the DEX and the CEX become more costly to trade, the reaction of β_{buy}^* and β_{sell}^* can be ambiguous. Figure 5 shows that the impact of DEX liquidity (X^*) dominates that of CEX liquidity (S; the bid-ask spread) for buying informed traders, leading to a decline in β_{buy}^*. In contrast, β_{sell}^* shows the opposite reaction to σ. Moreover, since β_{sell} exhibits a stronger (positive) reaction to σ than a negative reaction of β_{buy}, the DEX involves more informed trading in expectation, i.e., $\beta_{buy}^* + \beta_{sell}^*$ increases.

Numerical result 1: *When the asset becomes more volatile, informed sellers tend to cluster on the DEX and informed buyers tend to cluster on the CEX. The net effect is positive in the sense that outflow of buyers is dominated by inflow of sellers to the DEX.*

Figure 5: Reaction of informed traders to σ



Note: These figures are illustrated by using $z = 2.0$. The solid line shows β_{buy}, the dotted line shows β_{sell}, and the dashed line shows $\beta_{buy} + \beta_{sell}$.

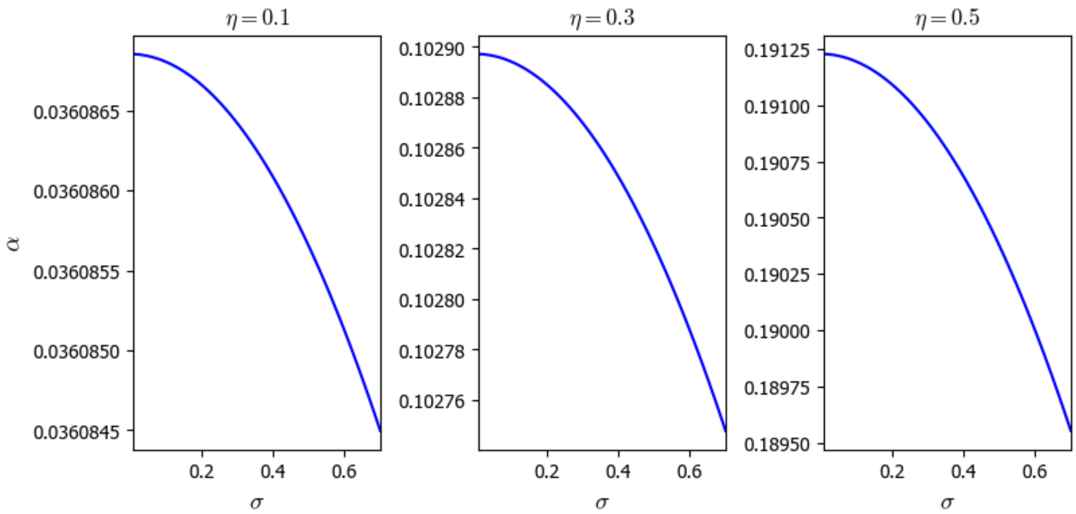
The above result is intuitive because the convexity of the CFMM makes it less costly for a liquidity taker to add liquidity to the pool than consume liquidity in the pool. The asymmetric price impact means that an incentive to migrate to (or stick to) the DEX is stronger for selling informed traders when the amount of DEX liquidity declines. As a result, informed traders on the DEX exhibit asymmetric reaction to a volatility shock: when selling (resp. buying) the asset, informed traders tend to cluster on the DEX (resp. the CEX).

Next, consider the behavior of liquidity traders. Figure 6 shows that liquidity traders tend to cluster on the CEX when the asset becomes more volatile. They compare the delay cost on the DEX ($\gamma\sigma$) to the expected trading cost on the CEX (S; the bid-ask spread). Since both of them are proportional to the asset volatility, σ has no direct impacts on liquidity traders' venue choice. Instead, what matters is the *normalized* bid-ask spread, $\frac{S}{2\sigma}$, which captures the adverse selection problem for the CEX market makers that emanates from informed traders' venue choice.

In the above discussion, we have established that informed traders tend to cluster on the DEX in expectation (i.e., $\frac{\beta_{buy}+\beta_{sell}}{2}$ increases), and it imposes more severe adverse selection on DEX market makers, while mitigating that for CEX market makers. It tightens the normalized bid-ask spread on the CEX and attracts liquidity traders to the CEX.

Numerical result 2: *When the asset becomes more volatile, liquidity traders tend to cluster on the CEX.*

Figure 6: Reaction of liquidity traders to σ



Note: These figures are illustrated by using $z = 2.0$.

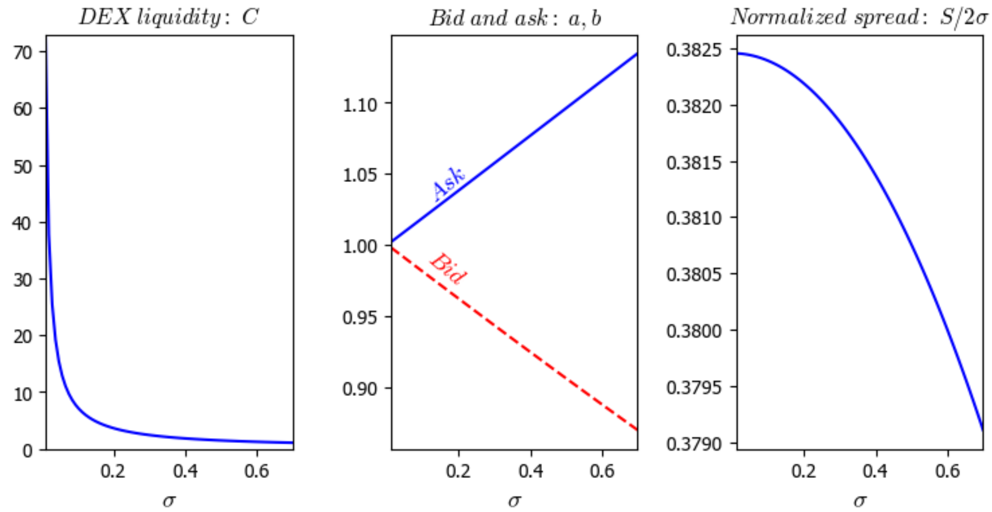
Finally, Figure 7 summarizes the reaction of market liquidity to a change in the asset volatility incorporating the above behavior of traders. Through their venue choice, traders' reactions may have indirect effects and undermine the direct impact of σ on market liquidity, but they cannot offset or dominate the direct effect.

Numerical result 3: *When the asset becomes more volatile, liquidity on the DEX, measured by the amount of cash locked in the pool, and liquidity on the CEX, measured by the bid-ask spread, both deteriorate. The normalized bid-ask spread on the CEX, however, improves.*

The normalized bid-ask spread becomes narrower because informed traders, in expectation, tends to cluster on the DEX, while liquidity traders are more likely to trade on the CEX.

DEX trading share. The share of the DEX in terms of trading volume can be used as a measure of traders' activity on the DEX. The expected trading volumes on the CEX and the DEX, as well as the

Figure 7: Reaction of market liquidity to σ



Note: These figures are illustrated by using $z = 2.0$ and $\eta = 0.3$. They are rbust to other parameter values as long as $\eta < \frac{C}{1+x_{passive}}$ is satisfied to guarantee the existence of the stable equilibrium.

aggregate volume, are defined by the following.

$$V_{CEX} = \eta \left(1 - \frac{1}{2} \sum_{i=buy,sell} \beta_i \right) + (1-\eta)(1-\alpha)z,$$

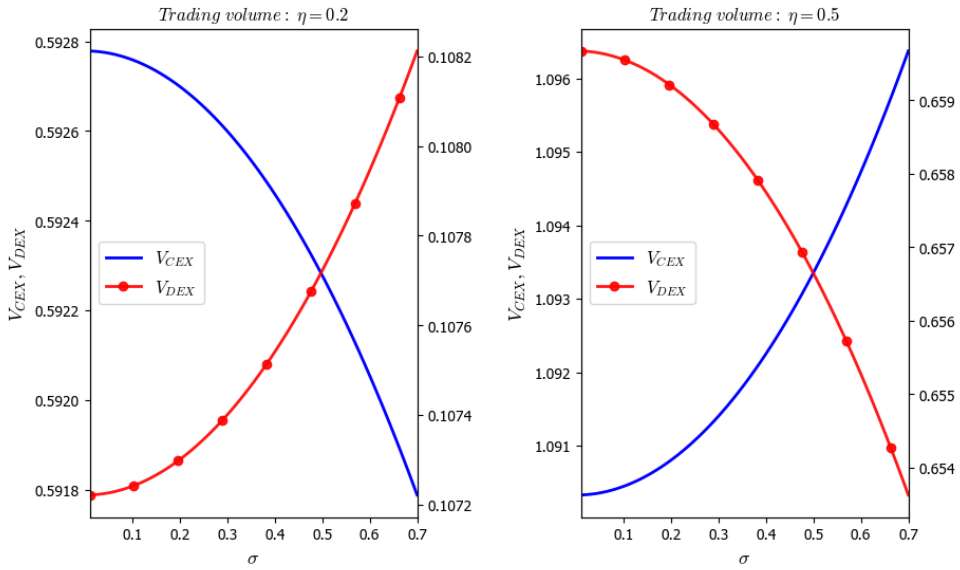
$$V_{DEX} = \eta \frac{1}{2} \sum_{i=buy,sell} \beta_i + (1-\eta)\alpha z,$$

$$V = V_{CEX} + V_{DEX} = \eta + (1-\eta)z.$$

Note that the aggregate trading volume, V, is perfectly determined by the probability of the trigger event (η) and the expected size of liquidity trading $z = \frac{1}{2}\mathbb{E}[z_{buy} + z_{sell}]$. So the question is how it is allocated across two exchanges.

Figure 8 plots the trading volumes on the DEX and the CEX (V_{DEX}, V_{DEX}) against the asset volatility. The behavior of trading volumes on exchanges are not robust and dependent on the probability of the trigger event η. When the asset becomes more volatile, the measure of DEX liquidity traders increases, while that of informed traders declines in expectation. From the above equations for V_i, these two effects compete against each other with η being the weight on the informed traders' behavior. Thus the DEX trading volume (as well as its share) tends to decrease when a common value shock is more likely to be the trigger of transactions (the right panel), while it tends to increase when

Figure 8: Trading volumes



Note: These figures are illustrated by using $z = 2.0, \eta = 0.3$ and $\sigma = 0.05$. They are rbust to other parameter values as long as $\eta < \frac{x_{passive}}{1+x_{passive}}$ is satisfied to guarantee the existence of the stable equilibrium.

transactions are motivated by private values.

5 Discussion

5.1 Empirical implications

We can derive novel empirical implications from our model. We consider a change in the asset volatility σ (or the degree of adverse selection) and the addition of a DEX with the CFMM to the traditional financial market as an exogenous variation in financial market to propose testable implications. More concretely, the addition of a DEX can be seen as a change in the status quo, in which all transactions are conducted via centralized exchanges. It is a relevant measure of the impact of DEXs because, in reality, many ERC-20 tokens are cross-listed between some CEXs and DEXs. Also, we have several tokens that replicate some major cryptocurrencies. For example, Uniswap has listed the ETH/WBTC pair on December 2020. WBTC is an ERC-20 token that is pegged to Bitcoin. Thus, the listing of WBTC on Uniswap can be seen as the advent of a DEX for the Ethereum and Bitcoin pair, which is previously traded mostly on the centralized exchanges, such as Coinbase and Binance.

Firstly, the equilibrium prices of the asset on the CEX is affected by an automated market.

Conjecture 1. *The addition of the DEX with the CFMM induces or strengthens asymmetry in the bid and the ask prices.*

The first conjecture is a natural consequence of the convex pricing of the CFMM. It generates asymmetry in the price impact on the DEX for buy and sell orders. The asymmetry, in turn, must affect bid and ask prices on the CEX with different magnitudes via traders' venue choice. There is a large body of literature for the asymmetric bid and ask prices, such as Ho and Stoll (1981) and Stoll (1989). In terms of the bid-ask spread that stems from adverse selection, studies have highlighted the asymmetry due to some microstructure constraints, such as a discrete tick size (Anshuman and Kalay, 1998), and the asymmetric distribution for the value of assets (Bossaerts and Hillion, 1991). Our model proposes a new market structure that brings about the asymmetric prices on the traditional limit-order markets.

Conjecture 2. *All else being equal, an increase in the asset volatility (or the degree of adverse selection) is associated with a higher order informativeness on the DEX and a lower order informativeness on the CEX.*

This conjecture is the result of Subsection 4.2. A higher degree of adverse selection makes it costly to trade on both venues. Informed traders tend to cluster on the same side of the market on the DEX, bearing a larger price impact, compared to liquidity traders with random trading behavior. Thus, order flow on the DEX tends to be information driven, while that on the CEX tends to be private-value driven.

As the literature on the CFMM is still in its infancy, an empirical measure of informativeness on the DEX is yet to be constructed. In contrast, we have some metrics of informed trading on the traditional markets, such as PIN by Easley and O'hara (1987). Our model suggests that the addition of a DEX with the CFMM strengthens a positive reaction of informativeness of order flow to a change in the volatility of the asset.

Related to the informativeness of order flow, our model suggests that buy and sell orders may react in the different ways even if the magnitude of a trigger event is the same. This, in turn, implies that return predictability of order flow is asymmetric between sell and buy orders.

Conjecture 3. *Buy orders on the DEX are more likely to be followed by a positive innovation in returns than sell orders followed by a negative innovation. The opposite is true on the CEX.*

Moreover, the above prediction regarding the informativeness of order flow has a direct implication for market liquidity.

Conjecture 4. *All else being equal, an increase in the asset volatility (or the degree of adverse selection) is associated with a decline in the amount of assets locked in the DEX (e.g., Uniswap), a wider effective bid-ask spread, and a narrower normalized bid-ask spread on the CEX.*

In terms of the above-mentioned example, our model predicts that the correlation between the effective bid-ask spread for the ETH/BTC pair on centralized exchanges (e.g., Coinbase) and its return volatility tends to be stronger after Uniswap starts covering the ETH/WBTC pair compared to that prior to Uniswap.

5.2 Limitations of the model

Information horizon. One of the limitations of our model is that it does not accommodate informed traders with a longer information horizon. In our model, we consider a one-shot trading game, in which informed traders try to exploit their informational advantage knowing that the information is perfectly revealed (or the asset is liquidated) after a trade.

When traders act on some long-lived private information, we need to incorporate some other important features of the information management on the DEX, namely, public nature of blockchain information. As mentioned in Subsection A, trading intentions on the DEX are stored in the mempool and wait for validation by blockchain miners. In most cases, the state of the mempool is publicly disseminated and observable for miners and traders. As suggested by Malinova and Park (2017), a trader may extract private information of other traders by observing publicly available information on the mempool, generating the front-running risk. Our current model with competitive traders cannot fully address the long-run issues, and a strategic aspect of informed trading must be embedded in the future research.

Endogenous delay. In our model, we cut corners in introducing a trading delay on the DEX by assuming that a liquidity trader incurs a linear delay cost per transaction. It can be thought of as a situation where the mass of liquidity trading is sufficiently small compared to the block capacity so that all trades are settled with a constant (and deterministic) delay, i.e., the block time.

In general, however, a trader can shorten the expected waiting time by paying a higher transaction fee to blockchain miners. Since a miner tends to process transactions with higher fees, proposing a higher fee can put trader's transaction forward in a queue. For example, Huberman et al. (2019) formulate the expected delay cost (the sum of the waiting time and fee payment) as an increasing function of the measure of traders waiting for verification.

Endogenizing the delay cost will certainly add new implications to our model. At the same time, however, we also believe that the endogenous delay cost strengthens our results. If the delay cost is an increasing function of the measure of liquidity traders on the DEX (α), as suggested by Huberman et al. (2019), liquidity traders are discouraged to participate in the DEX, leading to even larger outflow of liquidity traders from the DEX to the CEX. Thus, the endogenous delay cost may work as an additional driving force to mitigate adverse selection for CEX market makers and improve CEX market liquidity.

6 Conclusion

This paper studies the equilibrium impact of the adoption of a decentralized exchange (DEX) with a novel market-making algorithm called the Constant Function Market Makers (CFMMs). In the real financial market, DEXs with the CFMM and the traditional centralized exchanges (CEXs) with the limit order mechanism interact with each other. We construct a model to describe such a coexistence where traders are endogenously differentiated between the DEX and the CEX depending on their trading motives—informed or uninformed.

We first derive the equilibrium measure of informed and uninformed traders who trade on the DEX and the CEX. The model shows that the amount of liquidity locked in the DEX has a positive impact on the liquidity measure on the CEX, i.e., the bid-ask spread on the CEX shrinks when the DEX has more liquidity. We also characterize the profit function of market makers on the DEX who supply the asset and cash following the CFMM algorithm. As suggested by the literature, they face the impermanent loss. In addition, we show that they also have a chance to earn profits from noise trading, which is absent in the literature of automated markets. Thus even without fee rebates for market makers, they have an incentive to provide liquidity. Based on the derived profit function, we endogenize the amount of liquidity on the DEX and conduct some comparative statics.

In our model, we focus on a one-shot trading environment and abstract away from long-lived private information. When information horizon becomes longer, informed traders must incorporate the speed of information revelation via their trading orders (as in Kyle, 1985). Moreover, price discovery in the long-run is one of the two pillars that determines trader welfare. Thus, constructing a long-run model based on the current analyses is the topic for future research.

References

Abadi, Joseph and Markus Brunnermeier, "Blockchain economics," Technical Report, mimeo Princeton University 2018.

Adams, Hayden, Noah Zinsmeister, and Dan Robinson, "Uniswap v2 core," *URl: https://uniswap.org/whitepaper. pdf*, 2020.

Angeris, Guillermo, Alex Evans, and Tarun Chitra, "When does the tail wag the dog? Curvature and market making," *arXiv preprint arXiv:2012.08040*, 2020.

_ **and Tarun Chitra**, "Improved Price Oracles: Constant Function Market Makers," *arXiv preprint arXiv:2003.10001*, 2020.

_ **, Hsien-Tang Kao, Rei Chiang, Charlie Noyes, and Tarun Chitra**, "An analysis of Uniswap markets," *Cryptoeconomic Systems Journal*, 2019.

Anshuman, V Ravi and Avner Kalay, "Market making with discrete prices," *The Review of Financial Studies*, 1998, *11* (1), 81–109.

Antonopoulos, Andreas M, *Mastering Bitcoin: unlocking digital cryptocurrencies*, O'Reilly Media, Inc., 2014.

_ **and Gavin Wood**, *Mastering ethereum: building smart contracts and dapps*, O'reilly Media, 2018.

Bagehot, Walter, "The only game in town," *Financial Analysts Journal*, 1971, *27* (2), 12–14.

Black, Fischer, "Noise," *The journal of finance*, 1986, *41* (3), 528–543.

Bossaerts, Peter and Pierre Hillion, "Market Microstructure Effects of Government Intervention in the Foreign Exchange Market," *The Review of Financial Studies*, 1991, *4* (3), 513–541.

Brolley, Michael and David A Cimon, "Order-flow segmentation, liquidity, and price discovery: the role of latency delays," *Journal of Financial and Quantitative Analysis*, 2020, *55* (8), 2555–2587.

_ **and Marius Zoican**, "Liquid speed: A congestion fee for low-latency exchanges," *Rotman School of Management Working Paper*, 2020, (3377346).

Brunnermeier, Markus K and Lasse Heje Pedersen, "Market liquidity and funding liquidity," *The review of financial studies*, 2009, *22* (6), 2201–2238.

Chen, Long, Lin William Cong, and Yizhou Xiao, "A Brief Introduction to Blockchain Economics," *Available at SSRN*, 2019.

Chiu, Jonathan and Thorsten V Koeppl, "The economics of cryptocurrencies–bitcoin and beyond," *Available at SSRN 3048124*, 2017.

Cong, Lin William, Ye Li, and Neng Wang, "Tokenomics: Dynamic adoption and valuation," 2018.

Easley, David and Maureen O'hara, "Price, trade size, and information in securities markets," *Journal of Financial economics*, 1987, *19* (1), 69–90.

Evans, Alex, "Liquidity Provider Returns in Geometric Mean Markets," *arXiv preprint arXiv:2006.08806*, 2020.

Glosten, Lawrence R and Paul R Milgrom, "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders," *Journal of Financial Economics*, 1985, *14* (1), 71–100.

Grossman, Sanford J and Joseph E Stiglitz, "On the impossibility of informationally efficient markets," *The American economic review*, 1980, *70* (3), 393–408.

Hanson, Robin, "Combinatorial information market design," *Information Systems Frontiers*, 2003, *5* (1), 107–119.

Harvey, Campbell, "Cryptofinance," *Available at SSRN 2438299*, 2016.

Harvey, Campbell R., Ashwin Ramachandran, and Joseph Santoro, "DeFi and the Future of Finance," *SSRN Electronic Journal*, 2020.

Ho, Thomas and Hans R Stoll, "Optimal dealer pricing under transactions and return uncertainty," *Journal of Financial economics*, 1981, *9* (1), 47–73.

Huberman, Gur, Jacob Leshno, and Ciamac C Moallemi, "An economic analysis of the bitcoin payment system," *Columbia Business School Research Paper*, 2019, (17-92).

Kyle, Albert S, "Continuous auctions and insider trading," *Econometrica*, 1985, pp. 1315–1335.

Lee, Tomy, "Latency in fragmented markets," *Review of Economic Dynamics*, 2019, *33*, 128–153.

Lehar, Alfred and Christine A Parlour, "Liquidity Demand and Bitcoin Transaction Fees," Technical Report, Working paper 2019.

Li, Sida, Mao Ye, and Miles Zheng, "Financial Regulation, Clientele Segmentation, and Stock Exchange Order Types," Technical Report, National Bureau of Economic Research 2021.

Long, J Bradford De, Andrei Shleifer, Lawrence H Summers, and Robert J Waldmann, "Noise trader risk in financial markets," *Journal of political Economy*, 1990, *98* (4), 703–738.

Malinova, Katya and Andreas Park, "Market Design with Blockchain Technology," *Available at SSRN 2785626*, 2017.

Menkveld, Albert J and Marius A Zoican, "Need for speed? Exchange latency and liquidity," *The Review of Financial Studies*, 2017, *30* (4), 1188–1228.

Nakamoto, Satoshi, "Bitcoin: A peer-to-peer electronic cash system," 2008.

Pagano, Marco, "Trading volume and asset liquidity," *The Quarterly Journal of Economics*, 1989, *104* (2), 255–274.

Pagnotta, Emiliano and Andrea Buraschi, "An equilibrium valuation of bitcoin and decentralized network assets," *Available at SSRN 3142022*, 2018.

Saleh, Fahad, "Blockchain without waste: Proof-of-stake," 2018.

Schilling, Linda and Harald Uhlig, "Some Simple Bitcoin Economics," *NBER Working Paper*, 2018, *No. 24483.*

Stoll, Hans R, "Inferring the components of the bid-ask spread: Theory and empirical tests," *the Journal of Finance*, 1989, *44* (1), 115–134.

Szabo, Nick, "Formalizing and securing relationships on public networks," *First Monday*, 1997, *2* (9).

Warren, Will and Amir Bandeali, "0x: An open protocol for decentralized exchange on the Ethereum blockchain," *URl: https://github. com/0xProject/whitepaper*, 2017.

Ye, Linlin, "Understanding the impacts of dark pools on price discovery," *Available at SSRN 2874957*, 2016.

Ye, Mao, "A glimpse into the dark: Price formation, transaction cost and market share of the crossing network," *Transaction Cost and Market Share of the Crossing Network (June 9, 2011)*, 2011.

Zhang, Yi, Xiaohong Chen, and Daejun Park, "Formal specification of constant product (xy=k) market maker model and implementation," *manuscript*, 2018.

Zhu, Haoxiang, "Do dark pools harm price discovery?," *The Review of Financial Studies*, 2014, *27* (3), 747–789.

A Blockchain technology

The blockchain can be seen as a novel way of managing and tracking transactions information. In the traditional world, we typically maintain a ledger that records participants' state information in a centralized manner, e.g., a bank acts as an intermediary. Bilateral transactions with no intermediation by a credible third party incur asymmetric information and settlement risk.

In contrast, on the blockchain platform, a ledger is not held by a particular entity, but is distributed across all participants in the network, called record keepers or blockchain miners. The distributed ledger system requires information about blockchain users to be a consensus among all record keepers. This highlights its first difference from traditional transactions, in which only a centralized authority keeps track of information. Due to its distributed nature, the blockchain is robust to a single point of failure and does not incur costs of building credibility.

A transaction with a distributed record-keeping system by blockchain goes as follows. Suppose that Alice wants to buy a cup of coffee at Bob's cafe by paying Bitcoin. Information about this transaction must be validated by blockchain miners. More precisely, the transaction is added to a block by a miner. A sequence of blocks are encrypted and become a blockchain. In the Bitcoin blockchain, for example, each miner in the network maintains a temporary list of unconfirmed transactions, called a *mempool*. Transactions in the mempool are yet to be recorded on the blockchain, and information on the mempool is public to the network. A miner picks one of the transactions in the pool and tries to validate it by executing costly computation following a certain algorithm. The fastest miner who solves the problem adds transaction information to a block (i.e., she mines a block). The reward for mining a block is a fee: when Alice initiates a transaction, she attaches a fee to her transaction, and the validating miner obtains the attached fee.[17]

In general, it is extremely difficult for one miner in the network to overturn the consensus. In the case of Bitcoin or Ethereum, for example, they leverage their computing power to solve a time-consuming cryptographic problem. This process is called proof of work (PoW), and the miner who performs it fastest is entitled to add a new block a chain.[18] Of course there can be multiple chains of blocks, because each miner can choose to which blockchain she adds a newly mined block. Following Nakamoto (2008), however, the longest chain is regarded as a valid chain. Therefore, if a malicious agent attempts to add fraudulent information to the transaction history (e.g., a double-spending problem), she must outpace all miners in the network and secretly generate a longer chain than other chains, which requires prohibitively high computing power. That is, information on the blockchain is (almost) free from tampering.

Moreover, Ethereum allows users to add complex scripts to the blockchain which describe the conditions under which transaction is verified and recorded. It implies that a transaction takes place only if the conditions in the code are fulfilled, and it is done automatically without any centralized third-party agencies. This type of automated contracts are called a *smart contract* following Szabo (1997).

B Contingent venue choice by liquidity traders

In this appendix, we check the robustness of our results by relaxing the assumption regarding liquidity traders' venue choice. We allow liquidity traders to choose their trading venue contingent on the realized sign of a private-value shock. Due to the convexity of the CPMM pricing, we focus on the

[17]A miner also obtains a block reward, which is a constant amount of Bitcoin (or other cryptocurrency in other blockchains), when she mines a block. Although the block reward incentivizes miners to leverage their computing power, the amount of reward periodically shrinks and converges to zero in the future.

[18]There are several ways to reach a consensus, and different blockchains (including ETH 2.0) adopt different processes. For example, Saleh (2018) analyzes the viability of the proof of stake (PoS).

equilibrium in which the fractions of buying and selling liquidity traders on the DEX are asymmetric and given by $\alpha_{buy} \in (0, 1)$ and $\alpha_{sell} \in (0, 1)$, respectively.

By applying the same logic as the previous sections, informed traders' indifference conditions are given by

$$1 + a(\beta_{buy}, \alpha_{buy}) = p(\beta_{buy}), \tag{17}$$

$$1 - b(\beta_{sell}, \alpha_{sell}) = p(-\beta_{sell}), \tag{18}$$

where p is given by equation (10), and the ask and the bid prices are given by (1) and (2) with asymmetric α. As a result, the equilibrium measure of informed buyers and sellers can be expressed by reusing the previous equations.

Corollary 1. *Given $\alpha \equiv (\alpha_{buy}, \alpha_{sell})$, the equilibrium measures of informed buyers and sellers on the DEX, $(\beta^*_{buy}, \beta^*_{sell})$, solve the indifference conditions in (17) and (18). There exist a unique set of solutions and they are stable. β^*_i is increasing in X, σ, and α_i for $i \in \{buy, sell\}$.*

Thus, the reaction of informed traders in the partial equilibrium stays the same as the previous case with symmetric α in Proposition 1.

Now, consider the venue choice for liquidity traders. When a liquidity trader buys (resp. sells) the asset on the CEX, her trading cost (resp. reward) is the ask (resp. bid) price. In contrast, she pays or obtains the following symmetric price on the DEX:

$$p_{noise}(\alpha_{buy}, \alpha_{sell}) = \mathbb{E}_{(z_{buy}, z_{sell})}\left[p(\alpha_{buy} z_{buy} - \alpha_{sell} z_{sell})\right],$$

where $\mathbb{E}_{(z_{buy}, z_{sell})}$ is the expectation regarding $z_i \sim U[0, z]$. As an example, the CPMM generates the following explicit formula:

$$\begin{aligned} p_{noise}(\alpha_{buy}, \alpha_{sell}) &= \mathbb{E}_{(z_{buy}, z_{sell})}\left[\frac{X}{X - (\alpha_{buy} z_{buy} - \alpha_{sell} z_{sell})}\right] \\ &= \frac{X}{z^2 \alpha_{buy} \alpha_{sell}} \log \frac{(X + \alpha_{sell} z)^{X + \alpha_{sell} z}(X - \alpha_{buy} z)^{X - \alpha_{buy} z}}{X^X (X - z\Delta\alpha)^{X - z\Delta\alpha}}. \end{aligned} \tag{19}$$

When α is symmetric, the net expected amount of liquidity trading is zero, as z_{buy} and $-z_{sell}$ are symmetrically distributed, and buy and sell orders are netted out. In contrast, the asymmetric behavior of buy and sell liquidity traders prevents the orders from completely offsetting each other.

When deciding on the trading venue, a liquidity trader with delay cost γ compares the trading cost on the DEX (the LHS) and the CEX (the RHS):

$$\gamma\sigma \gtrless \begin{cases} 1 + a(\beta_{buy}, \alpha_{buy}) - p_{noise}(\alpha_{buy}, \alpha_{sell}) & \text{if a "buy" liquidity shock hits,} \\ p_{noise}(\alpha_{buy}, \alpha_{sell}) - (1 - b(\beta_{sell}, \alpha_{sell})) & \text{if a "sell" liquidity shock hits.} \end{cases}$$

Since γ uniformly distributes over $[0, 1]$, we obtain the following:

Corollary 2. *Given $(\beta_{buy}, \beta_{sell})$, the equilibrium measures of liquidity buyers and sellers on the DEX are given by the solution of the following equations.*

$$\alpha_{buy} = \frac{1 + a(\beta_{buy}, \alpha_{buy}) - p_{noise}(\alpha_{buy}, \alpha_{sell})}{\sigma},$$

$$\alpha_{sell} = \frac{p_{noise}(\alpha_{buy}, \alpha_{sell}) - (1 - b(\beta_{sell}, \alpha_{sell}))}{\sigma}.$$

For $i \in \{buy, sell\}$, α_i is decreasing in β_i and increasing in α_j for $j \neq i$.

The above result shows that the reaction of α_i in the partial equilibrium is the same as the previous analyses. The additional result brought by the asymmetric α is the strategic complementarity between liquidity buyers and sellers. Namely, liquidity buyers are more willing to trade on the DEX when more liquidity sellers participate in the DEX, and vice versa. This is because a larger trading volume on the opposite side of the market offsets the buy liquidity orders, leading to a smaller shift in the liquidity pools and a weaker price impact. Therefore, liquidity begets liquidity on the DEX, as in the traditional limit order markets (e.g., Pagano, 1989).

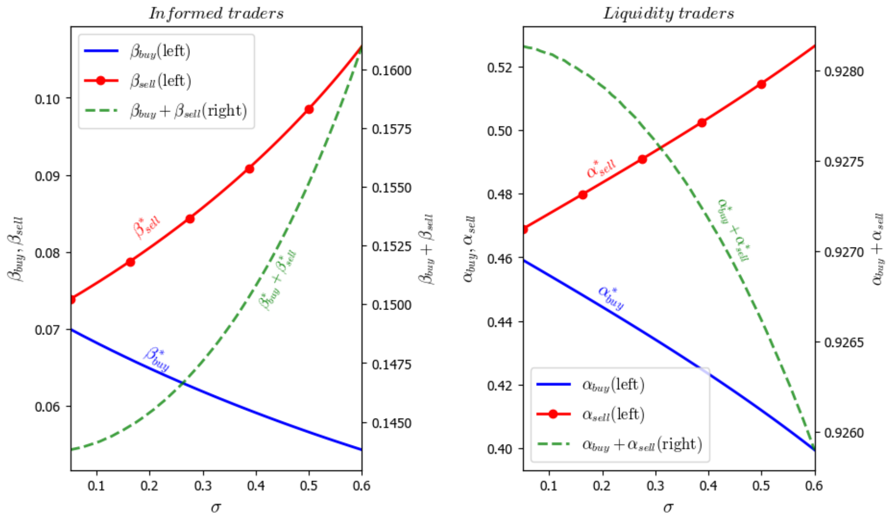
Finally, the expected profits for a market maker on the DEX is given by

$$\pi_M^D(X) = \frac{w\eta}{2}\left[\left(p(\beta_{buy}^*) - (1+\sigma)\right)\beta_{buy}^* + \left((1-\sigma) - p(-\beta_{sell}^*)\right)\beta_{sell}^*\right] \tag{20}$$

$$+ w(1-\eta)\mathbb{E}[p_{noise}(\alpha_{buy}, \alpha_{sell})(\alpha_{buy}z_{buy} - \alpha_{sell}z_{sell})] \tag{21}$$

Once again, it is easy to check that $\pi_{M,IT}^D < 0$ and $\pi_{M,LT}^D > 0$, meaning that a market maker loses from informed trading and gains from liquidity trading. A larger mass of informed trading on the DEX, as well as a higher volatility of the asset, reduces DEX market makers' profits by worsening adverse

Figure 9: Reaction of informed and liquidity traders



Note: This figures are illustrated by using $z = 2.0$ and $\eta = 0.3$. They are robust to other parameter values, as long as it holds that $z < \frac{C}{1+C}$.

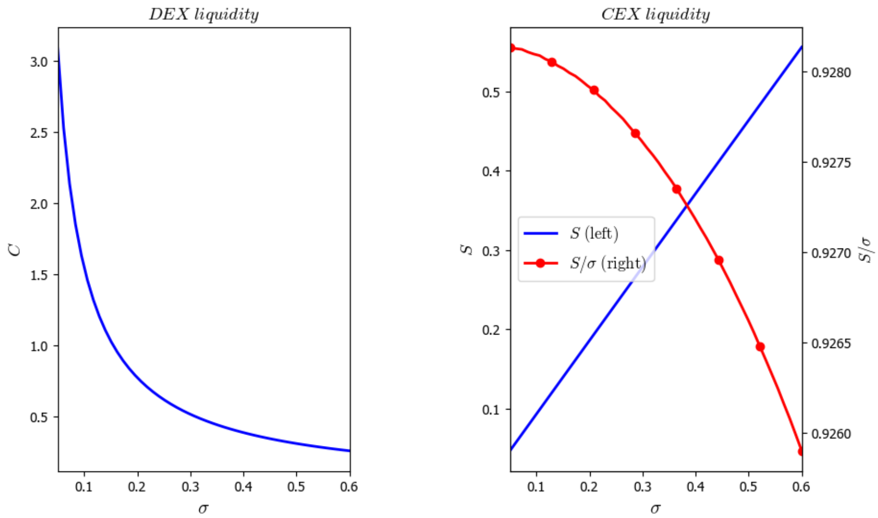
selection.

Numerical result. In what follows, we take the CPMM as an example to see the robustness. Figure 9 plots the reaction of informed traders (the left panel) and liquidity traders (the right panel) on the DEX to an increase in the volatility of the asset. The asymmetric reaction of informed buyers and sellers on the left panel shows that the result in the previous analyses is robust to a change in the assumption on liquidity traders' venue choice. The right panel, however, shows that allowing a contingent venue choice adds a new implication regarding liquidity traders' behavior on the DEX.

Numerical result 4: *When the asset becomes more volatile, liquidity buyers tend to cluster on the CEX, while liquidity sellers tend to cluster on the DEX. The net effect is negative, i.e., outflow of liquidity traders from the DEX dominates inflow to the DEX.*

The net behavior of liquidity traders $\alpha_{buy} + \alpha_{sell}$ is different from that of informed traders. Intuitively, a liquidity trader on the DEX is not directly affected by the convexity of the CPMM algorithm *per se*, as she is uncertain about the aggregate trading volume (given by [19]). Thus, the asymmetric reaction of liquidity traders is driven by the asymmetric reaction of informed traders, that is, β_{buy} and β_{sell}. Since the expected mass of informed traders increases on the DEX, the bid-ask spread on the CEX shrinks which, in turn, induces liquidity traders to participate more on the CEX in expectation. Therefore,

Figure 10: Reaction of market liquidity



Note: This figures are illustrated by using $z = 2.0$ and $\eta = 0.3$. They are robust to other parameter values, as long as it holds that $z < \frac{C}{1+C}$.

$\alpha_{buy} + \alpha_{sell}$ declines with σ.

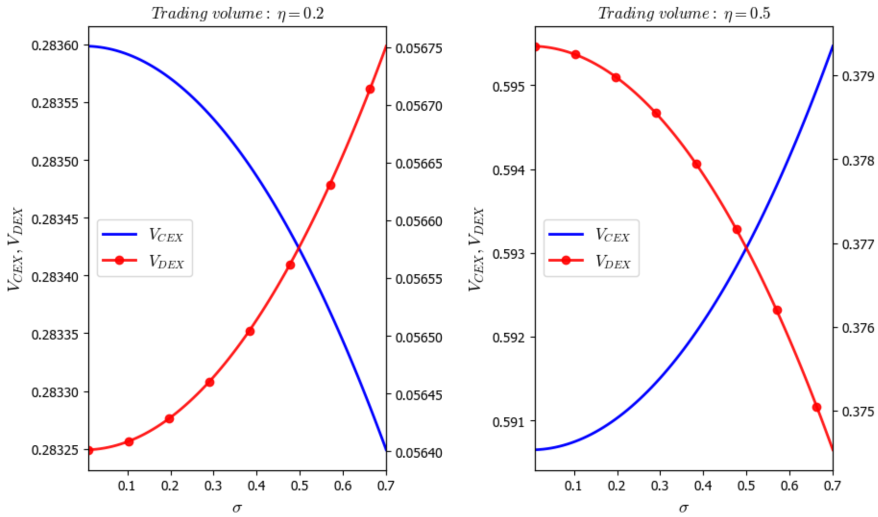
Given the venue choice by traders, Figure 10 shows the reactions of market liquidity. The left panel shows the comparative static of DEX liquidity, measured by X^*, while the right panel illustrates the bid-ask spread (S) and the normalized bid-ask spread (S/σ). Since the *net* behavior of liquidity traders stays the same as the previous sections, so does the impact of the asset volatility on market liquidity.

Finally, the reaction of trading volumes is similar to the case of symmetric α. When transactions are more likely to be driven by the common-value shock (resp. the private-value shock), the share of the DEX declines (resp. increases), as shown by Figure 11. The same logic applies as the previous sections, as the expected measure of liquidity traders ($\alpha_{buy} + \alpha_{sell}$) still declines even if we consider a state-contingent venue choice of liquidity traders. Thus, we can check that the ambiguous behavior of trading volumes provided in the previous section is robust.

C Proofs

We assume that CFMM function f satisfies the following regularity conditions. All conditions lead to the pricing function in Lemma 1 which provides intuition for the regularity conditions.

Figure 11: Reaction of trading volumes



Note: This figures are illustrated by using $z = 2.0$ and $\eta = 0.3$. They are robust to other parameter values, as long as it holds that $z < \frac{C}{1+C}$.

Condition 1. *The CFMM function $f : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ with initial liquidity pool (C, X) satisfies the following:*

(i) f is continuously differentiable, $\frac{\partial^2 f}{\partial c \partial x}$ exists, and $f_c(c, x)$ and $f_x(c, x)$ are positive for all $c, x > 0$;

(ii) If (c, x) is on function f, there exist $h : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $c = h(x; C, X)$ and h is decreasing in x;

(iii) $f_c(C, X) = f_x(C, X)$ holds for at the initial point (C, X);

(iv) $\{(c, x)|f(c, x) > k\}$ is a strictly convex set for all $k > 0$;

(v) $\frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)}$ is decreasing in X if and only if $\delta > 0$, convex in δ, and differentiable with respect to δ and X. Moreover, $\partial \frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)} / \partial X$ is decreasing in δ ;

(vi) $a(\delta, X) \equiv |\frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)} - 1|$ is log-submodular in (X, δ).

C.1 Proof of Proposition 1

By Condition 1-(iv), $\frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)}$ is increasing in δ and thus p is increasing in δ. By Condition 1-(v), it can be shown that p is decreasing in X if and only if $\delta > 0$. Since $\frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)}$ is differentiable, p is differentiable with respect to X and it is obvious that p is also differentiable with respect to δ.

Next, observe that

$$p_\delta(\delta, X) = \frac{1}{\delta} \left(\frac{f_x(h(X-\delta; C, X), X-\delta)}{f_c(h(X-\delta; C, X), X-\delta)} - p(\delta, X) \right).$$

Since $\frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)}$ is increasing and convex in δ, $\frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)} - p(\delta, X) > \frac{f_x(h(X+\delta;C,X),X+\delta)}{f_c(h(X+\delta;C,X),X+\delta)} - p(-\delta, X)$ holds. Hence, $p_\delta(b, X) > p_\delta(-b, X)$ holds for all $b, X > 0$. Next, we have

$$p_{\delta,X}(\delta, X) = \frac{1}{\delta}\left(\partial\frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)}/\partial X - p_X(\delta, X)\right).$$

Since $p_X(\delta, X) = \frac{1}{\delta}\int_0^\delta \partial\frac{f_x(h(X-\tilde{\delta}),X-\tilde{\delta})}{f_c(h(X-\tilde{\delta}),X-\tilde{\delta})}/\partial X d\tilde{\delta}$, Condition 1-(v) implies that $\partial\frac{f_x(h(X-\delta;C,X),X-\delta)}{f_c(h(X-\delta;C,X),X-\delta)}/\partial X < p_X(\delta, X)$ holds if and only if $\delta > 0$. Hence, $\frac{\partial^2 p}{\partial\delta\partial X} < 0$.

Suppose that $\delta' > \delta$ and $X' > X$. By Condition 1-(vi), $\frac{a(\delta,X)}{a(\delta,X')} < \frac{a(\delta',X)}{a(\delta',X')}$ holds. First, consider the case where $\delta' > \delta > 0$. Since

$$\frac{\int_0^\delta \frac{f_x(h(X-\tilde{\delta}),X-\tilde{\delta})}{f_c(h(X-\tilde{\delta}),X-\tilde{\delta})} - 1 d\tilde{\delta}}{\int_0^\delta \frac{f_x(h(X'-\tilde{\delta}),X'-\tilde{\delta})}{f_c(h(X'-\tilde{\delta}),X'-\tilde{\delta})} - 1 d\tilde{\delta}} < \frac{\int_\delta^{\delta'} \frac{f_x(h(X-\tilde{\delta}),X-\tilde{\delta})}{f_c(h(X-\tilde{\delta}),X-\tilde{\delta})} - 1 d\tilde{\delta}}{\int_\delta^{\delta'} \frac{f_x(h(X'-\tilde{\delta}),X'-\tilde{\delta})}{f_c(h(X'-\tilde{\delta}),X'-\tilde{\delta})} - 1 d\tilde{\delta}}$$

holds, $\frac{A(\delta,X)}{A(\delta,X')} < \frac{A(\delta',X)}{A(\delta',X')}$ must hold. Similar arguments hold for cases where $\delta' > 0 > \delta$and $0 > \delta' > \delta$. Thus, we obtain the result that $A(\delta, X) = |p(\delta, X) - 1|$ is submodular in (δ, X).

C.2 Proof of Proposition 2

Equation (14) is

$$\alpha = \frac{S(\beta_{buy}, \beta_{sell}, \alpha)/\sigma}{2} = \frac{1}{2}\left[\frac{(1-\beta_{buy})\eta}{(1-\beta_{buy})\eta + z(1-\alpha)(1-\eta)} + \frac{(1-\beta_{sell})\eta}{(1-\beta_{sell})\eta + z(1-\alpha)(1-\eta)}\right]. \quad (22)$$

It holds that $S(\beta_{buy}, \beta_{sell}, 1) = 2\sigma$. Thus, the above equation has $\alpha = 1$ as a solution. Also, from the indifference conditions for informed traders, $0 < \beta_i < 1$ for all $\alpha \in [0, 1]$. Now, observe that (22) is equivalent to:

$$1 - \alpha = \frac{z(1-\alpha)(1-\eta)}{2}\left[\frac{1}{(1-\beta_{buy})\eta + z(1-\alpha)(1-\eta)} + \frac{1}{(1-\beta_{sell})\eta + z(1-\alpha)(1-\eta)}\right].$$

When $\alpha \neq 1$,

$$1 = \frac{z(1-\eta)}{2}\left[\frac{1}{(1-\beta_{buy})\eta + z(1-\alpha)(1-\eta)} + \frac{1}{(1-\beta_{sell})\eta + z(1-\alpha)(1-\eta)}\right] \quad (23)$$

holds. Let the right hand side of this equation be $g(\alpha)$. Then, $g(0) < 1$. Moreover,

$$g'(\alpha) = \frac{z(1-\eta)}{2}\left[\frac{z(1-\eta)}{((1-\beta_{buy})\eta + z(1-\alpha)(1-\eta))^2} + \frac{z(1-\eta)}{((1-\beta_{sell})\eta + z(1-\alpha)(1-\eta))^2}\right] > 0$$

holds. Hence, an interior solution α^* exists if and only if $g(1) > 1$, which is equivalent to $z > z^*$. This also shows that α^* is unique if it exists. Stability of α^*follows from the fact that S is convex in α. The negative impact of β_i on α in the partial equilibrium is straightforward.

C.3 Proof of Proposition 3

Equation (23) implies:

$$0 = \frac{\frac{\partial \beta_{buy}}{\partial X}\eta + \frac{\partial \alpha}{\partial X}z(1-\eta)}{((1-\beta_{buy})\eta + z(1-\alpha)(1-\eta))^2} + \frac{\frac{\partial \beta_{sell}}{\partial X}\eta + \frac{\partial \alpha}{\partial X}z(1-\eta)}{((1-\beta_{sell})\eta + z(1-\alpha)(1-\eta))^2}$$

holds. By the conditions derived in Proposition 1, observe that:

$$\begin{aligned}-(1+\sigma)\eta\frac{\partial \beta^*_{buy}}{\partial X} - z(1-\eta)\frac{\partial \alpha}{\partial X} = \quad & (p_X(\beta^*_{buy}, X) + p_\beta(\beta^*_{buy}, X)\frac{\partial \beta^*_{buy}}{\partial X})((1-\beta^*_{buy})\eta + z(1-\alpha)(1-\eta)) \\ & -p(\beta^*_{buy}, X)\left(\eta\frac{\partial \beta^*_{buy}}{\partial X} + z(1-\eta)\frac{\partial \alpha}{\partial X}\right)\end{aligned}$$

and this is equivalent to:

$$\begin{aligned}&((1+\sigma - p(\beta^*_{buy}, X))\eta + p_\beta(\beta^*_{buy}, X)((1-\beta^*_{buy})\eta + z(1-\alpha)(1-\eta)))\frac{\partial \beta^*_{buy}}{\partial X} \\ &+ p_X(\beta^*_{buy}, X)((1-\beta^*_{buy})\eta + z(1-\alpha)(1-\eta)) \\ &= (p(\beta^*_{buy}, X) - 1)z(1-\eta)\frac{\partial \alpha}{\partial X}\end{aligned}$$

Similarly, observe that:

$$\begin{aligned}-(1-\sigma)\eta\frac{\partial \beta^*_{sell}}{\partial X} - z(1-\eta)\frac{\partial \alpha}{\partial X} = \quad & (p_X(-\beta^*_{sell}, X) - p_\beta(-\beta^*_{sell}, X)\frac{\partial \beta^*_{sell}}{\partial X})((1-\beta^*_{sell})\eta + z(1-\alpha)(1-\eta)) \\ & -p(-\beta^*_{sell}, X)\left(\eta\frac{\partial \beta^*_{sell}}{\partial X} + z(1-\eta)\frac{\partial \alpha}{\partial X}\right)\end{aligned}$$

and this is equivalent to:

$$
\begin{aligned}
&((p(-\beta^*_{sell}, X) - (1-\sigma))\eta + p_\beta(-\beta^*_{sell}, X)((1-\beta^*_{sell})\eta + z(1-\alpha)(1-\eta)))\frac{\partial \beta^*_{sell}}{\partial X} \\
&- p_X(-\beta^*_{sell}, X)((1-\beta^*_{sell})\eta + z(1-\alpha)(1-\eta)) \\
&= (1 - p(-\beta^*_{sell}, X))z(1-\eta)\frac{\partial \alpha}{\partial X}
\end{aligned}
$$

Combining these results, we obtain $\frac{\partial \alpha}{\partial X} < 0$ since p_X is negative (positive) when its first argument is positive (negative). Furthermore, at lease one of $\frac{\partial \beta^*_i}{\partial C_2}$ is positive.

We may further rewrite the above equations as:

$$
\begin{aligned}
A\frac{\partial \beta^*_{buy}}{\partial X} &= z(1-\eta)\frac{\partial \alpha}{\partial X} - \frac{(1-\beta^*_{buy})\eta + z(1-\alpha)(1-\eta)}{p(\beta^*_{buy}, X) - 1} p_X(\beta^*_{buy}, X) \\
B\frac{\partial \beta^*_{sell}}{\partial X} &= z(1-\eta)\frac{\partial \alpha}{\partial X} + \frac{(1-\beta^*_{sell})\eta + z(1-\alpha)(1-\eta)}{1 - p(-\beta^*_{sell}, X)} p_X(-\beta^*_{sell}, X)
\end{aligned}
$$

for some positive A, B. Lemma 1 Condition (vi) implies:

$$
\frac{p_X(\beta^*_{buy}, X)}{p(\beta^*_{buy}, X) - 1} < \frac{-p_X(-\beta^*_{sell}, X)}{1 - p(\beta^*_{sell}, X)} < 0
$$

Hence, we obtain:

$$
A\frac{\partial \beta^*_{buy}}{\partial X} > B\frac{\partial \beta^*_{sell}}{\partial X}
$$

Since at lease one of $\frac{\partial \beta^*_i}{\partial C_2}$ is positive, $\frac{\partial \beta^*_{buy}}{\partial X}$ is positive. If $\frac{\partial \beta^*_{sell}}{\partial X} \geq 0$, we are done. Suppose $\frac{\partial \beta^*_{sell}}{\partial X} < 0$. Then,

$$
\begin{aligned}
0 &= \frac{\frac{\partial \beta_{buy}}{\partial X}\eta + \frac{\partial \alpha}{\partial X}z(1-\eta)}{((1-\beta_{buy})\eta + z(1-\alpha)(1-\eta))^2} + \frac{\frac{\partial \beta_{sell}}{\partial X}\eta + \frac{\partial \alpha}{\partial X}z(1-\eta)}{((1-\beta_{sell})\eta + z(1-\alpha)(1-\eta))^2} \\
&< \frac{\frac{\partial \beta_{buy}}{\partial X}\eta + \frac{\partial \alpha}{\partial X}z(1-\eta)}{((1-\beta_{sell})\eta + z(1-\alpha)(1-\eta))^2} + \frac{\frac{\partial \beta_{sell}}{\partial X}\eta + \frac{\partial \alpha}{\partial X}z(1-\eta)}{((1-\beta_{sell})\eta + z(1-\alpha)(1-\eta))^2}
\end{aligned}
$$

holds. Hence, $\frac{\partial \beta^*_{buy}}{\partial X} + \frac{\partial \beta^*_{sell}}{\partial X} > 0$ since $\frac{\partial \alpha}{\partial X} < 0$.

C.4 Proof of Proposition 4

Since $p(\beta^*_{buy}) < (1+\sigma)$ and $p(\beta^*_{sell}) > (1-\sigma)$, $\pi^D_{IT} < 0$ holds. First, let us prove that $\frac{\partial \alpha^*}{\partial \sigma} < 0$ in an equilibrium in which X is exogenously given but other variables are determined endogenously. Similar to the proof of Proposition 3, we get:

$$0 = \frac{\frac{\partial \beta_{buy}}{\partial \sigma}\eta + \frac{\partial \alpha}{\partial \sigma} z(1-\eta)}{((1-\beta_{buy})\eta + z(1-\alpha)(1-\eta))^2} + \frac{\frac{\partial \beta_{sell}}{\partial \sigma}\eta + \frac{\partial \alpha}{\partial \sigma} z(1-\eta)}{((1-\beta_{sell})\eta + z(1-\alpha)(1-\eta))^2}.$$

Next, consider the indifference condition of the informed traders. This implies:

$$\begin{aligned}(1-\beta^*_{buy})\eta - (1+\sigma)\eta\frac{\partial \beta^*_{buy}}{\partial \sigma} - z(1-\eta)\frac{\partial \alpha}{\partial \sigma} = \; & p_\beta(\beta^*_{buy}, C_2)\frac{\partial \beta^*_{buy}}{\partial \sigma}((1-\beta^*_{buy})\eta + z(1-\alpha)(1-\eta)) \\ & - p(\beta^*_{buy}, C_2)\left(\eta\frac{\partial \beta^*_{buy}}{\partial \sigma} + z(1-\eta)\frac{\partial \alpha}{\partial \sigma}\right)\end{aligned}$$

Similarly,

$$\begin{aligned}-(1-\beta^*_{sell})\eta - (1-\sigma)\eta\frac{\partial \beta^*_{sell}}{\partial \sigma} - z(1-\eta)\frac{\partial \alpha}{\partial \sigma} = \; & -p_\beta(-\beta^*_{sell}, C_2)\frac{\partial \beta^*_{sell}}{\partial \sigma}((1-\beta^*_{sell})\eta + z(1-\alpha)(1-\eta)) \\ & - p(-\beta^*_{sell}, C_2)\left(\eta\frac{\partial \beta^*_{sell}}{\partial \sigma} + z(1-\eta)\frac{\partial \alpha}{\partial \sigma}\right)\end{aligned}$$

These conditions imply:

$$A\frac{\partial \beta^*_{buy}}{\partial \sigma} = \frac{1}{\sigma}((1-\beta^*_{buy})\eta + z(1-\alpha)(1-\eta)) + z(1-\eta)\frac{\partial \alpha}{\partial \sigma}$$

$$B\frac{\partial \beta^*_{sell}}{\partial \sigma} = \frac{1}{\sigma}((1-\beta^*_{sell})\eta + z(1-\alpha)(1-\eta)) + z(1-\eta)\frac{\partial \alpha}{\partial \sigma}$$

for some $A, B > 0$. If $\frac{\partial \alpha}{\partial \sigma} \geq 0$, then $\frac{\partial \beta_{sell}}{\partial \sigma}, \frac{\partial \beta_{buy}}{\partial \sigma} > 0$. This leads to a contradiction. Hence, $\frac{\partial \alpha}{\partial \sigma} < 0$.

Moreover, Equation (23) implies:

$$0 = \frac{\frac{\partial \beta_{buy}}{\partial \sigma}\eta + \frac{\partial \alpha}{\partial \sigma} z(1-\eta)}{((1-\beta_{buy})\eta + z(1-\alpha)(1-\eta))^2} + \frac{\frac{\partial \beta_{sell}}{\partial \sigma}\eta + \frac{\partial \alpha}{\partial \sigma} z(1-\eta)}{((1-\beta_{sell})\eta + z(1-\alpha)(1-\eta))^2}.$$

Observe that:

$$\frac{2}{w}\pi^D_{IT} = -\frac{\sigma z(1-\alpha)(1-\eta)\beta_{buy}}{(1-\beta_{buy})\eta + z(1-\alpha)(1-\eta)} - \frac{\sigma z(1-\alpha)(1-\eta)\beta_{sell}}{(1-\beta_{sell})\eta + z(1-\alpha)(1-\eta)}$$

and thus

$$\frac{d(\frac{2}{w}\pi_{IT}^{D})}{d\sigma} = \frac{\pi_{IT}^{D}}{\sigma} - \sigma\frac{\frac{\partial\beta_{buy}}{\partial\sigma}(\eta+z(1-\alpha)(1-\eta)) - \frac{\partial\alpha}{\partial\sigma}z(1-\eta)\beta_{buy}}{((1-\beta_{buy})\eta+z(1-\alpha)(1-\eta))^2} - \sigma\frac{\frac{\partial\beta_{buy}}{\partial\sigma}(\eta+z(1-\alpha)(1-\eta)) - \frac{\partial\alpha}{\partial\sigma}z(1-\eta)\beta_{sell}}{((1-\beta_{sell})\eta+z(1-\alpha)(1-\eta))^2}$$

$$= \frac{\pi_{IT}^{D}}{\sigma} + \sigma\frac{\frac{\partial\alpha}{\partial\sigma}(z(1-\eta)\beta_{buy} + \frac{z(1-\eta)}{\eta}(\eta+z(1-\alpha)(1-\eta)))}{((1-\beta_{buy})\eta+z(1-\alpha)(1-\eta))^2} + \sigma\frac{\frac{\partial\alpha}{\partial\sigma}(z(1-\eta)\beta_{sell} + \frac{z(1-\eta)}{\eta}(\eta+z(1-\alpha)(1}{((1-\beta_{sell})\eta+z(1-\alpha)(1-\eta))^2}$$

$$< 0.$$

C.5 Proof of Proposition 5

Since Q is symmetric, we may rewrite the profit function as:

$$\pi_{LT}^{D} = w\int_{0}^{z}(p(\alpha\Delta z) - p(-\alpha\Delta z))\alpha\Delta z dQ(\Delta z).$$

This is positive because p is increasing in δ. Let us consider the comparative statics with respect to α. Observe that

$$(p(\alpha\Delta z) - p(-\alpha\Delta z))\alpha\Delta z = \int_{0}^{\alpha\Delta z}\frac{f_X(h(X-\delta),X-\delta)}{f_C(h(X-\delta),X-\delta)} - \frac{f_X(h(X+\delta),X+\delta)}{f_C(h(X+\delta),X+\delta)}d\delta$$

Since $\frac{f_X(h(X-\delta),X-\delta)}{f_C(h(X-\delta),X-\delta)}$ is increasing in δ, this object is increasing in α. Hence, π_{LT}^{D} is increasing in α,and thus decreasing in σ. Furthermore, since $\frac{f_X(h(X-\delta),X-\delta)}{f_C(h(X-\delta),X-\delta)}$ is increasing in X if and only if $\delta > 0$, this object is decreasing in X. Combined with the fact that w is decreasing in X, π_{LT}^{D} is decreasing in X.

C.6 Proof of Proposition 6

First, since $\left(p(\beta_{buy}^{*}) - (1+\sigma)\right)\beta_{buy}^{*} + \left((1-\sigma) - p(-\beta_{sell}^{*})\right)\beta_{sell}^{*}$, $\mathbb{E}[p(\alpha\Delta z)\alpha\Delta z]$ and w are decreasing in X, $\pi_{M}^{D}$is also decreasing in X for $\pi_{M}^{D} \geq 0$ given β_i^{*} and α^{*}. Next, for $\pi_{M}^{D} = 0$ to hold, we need $\frac{\eta}{2}\left[\left(p(\beta_{buy}^{*}) - (1+\sigma)\right)\beta_{buy}^{*} + \left((1-\sigma) - p(-\beta_{sell}^{*})\right)\beta_{sell}^{*}\right] + (1-\eta)\mathbb{E}[p(\alpha\Delta z)\alpha\Delta z] = 0$. Since LHS is decreasing in X , positive for sufficiently small X, and negative for sufficiently large X, there is a unique and stable solution X^{*}.

As we have shown in Proposition 5, π_{LT}^{D} is increasing in α. Hence, X^{*} is increasing in α. Now, observe that

$$\frac{2}{w\eta}\frac{\partial\pi_{IT}^{D}}{\partial\beta_{buy}^{*}} = \frac{f_x(h(X-\beta_{buy}^{*}),X-\beta_{buy}^{*})}{f_c(h(X-\beta_{buy}^{*}),X-\beta_{buy}^{*})} - (1+\sigma).$$

When $X > 1 + \frac{1+\sqrt{1+\sigma}}{\sigma}$, $\frac{\partial \pi_{IT}^{D}}{\partial \beta_{buy}^{*}} < 0$ holds. Hence, X^* is decreasing in β_{buy}^{*} as long as there is active liquidity supply. We can also show that X^* is decreasing in β_{sell}^{*} following a similar discussion.

D Sequential execution of orders at DEX

In the model, we assume that all market orders arriving at the DEX are simultaneously executed all at once. In this Appendix, we show that executing all at once (AAO) is the same as the sequential order execution.

Equivalence of post-trade liquidity pools Suppose that there are n informed traders, and each of them has measure $w = \frac{1}{n}$ and places δ unit of market buy order to the DEX (in the model, we assume $\delta = 1$). Note that the aggregate trading is of size δ. The initial state of the liquidity pool is denoted as (C_0, X_0) with $k \equiv C_0 X_0$. Note that the following discussion can be easily extended to the case with liquidity traders.

The first transaction is executed at price

$$p_1 = \frac{C_0}{X_0 - \delta w}$$

and the liquidity pool becomes

$$\begin{aligned} C_1 &= C_0 + p_1 \delta w = C_0 \frac{X_0}{X_0 - \delta w}, \\ X_1 &= X_0 - \delta w. \end{aligned}$$

By iterating, we obtain the following transition equations for the liquidity pools: for general $i = 1, 2, \cdots, n$,

$$\begin{aligned} C_i &= C_{i-1} \frac{X_{i-1}}{X_{i-1} - \delta w}, \\ X_i &= X_{i-1} - \delta w. \end{aligned}$$

The above equations imply that, after all (n) transactions are completed, the liquidity pools have

$$X_n = X_0 - n\delta w = X_0 - \delta,$$
$$C_n = C_0 \frac{X_0}{X_0 - n\delta w} = C_0 \frac{X_0}{X_0 - \delta}.$$

Thus, the post-trade state of the pools with sequential execution is the same as that of AAO execution. The above result also implies that the profits for the market makers on the DEX stay the same even if we consider sequential execution of orders.

Equivalence of the execution price Next, consider the expected trading cost (i.e., the execution price) for an informed trader. We consider a continuum of traders with measure β (by setting $n \to \infty$ with $\delta = 1$ and $w = \beta/n$ in the above example) and assume that traders' orders are independently executed following a Poisson process. Suppose that $y \in [0, \beta)$ orders have been executed before an informed trader gets to execute her order. From the above discussion, her order faces the following liquidity pools.

$$C_y = C_0 \frac{X_0}{X_0 - y}, \; X_y = X_0 - y.$$

Since her order is infinitesimal, it is executed at price

$$p(y) = \frac{C_y}{X_y} = \frac{C_0 X_0}{(X_0 - y)^2}.$$

Due to the independent Poisson process, $y \sim U[0, \beta]$. Thus, the expected execution price is given by

$$p = \frac{1}{\beta} \int_0^\beta p(y) dy = \frac{C_0}{X_0 - \beta},$$

which is identical to the execution price of each order in the case with AAO trade execution.